Constrained quantum dynamics

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With thanks to all my collaborators

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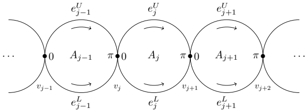
- In infinite graphs even an *Aharonov-Bohm field* that vanishes everywhere except one point may alter the spectrum dramatically.
- We will also mention a model of *soft waveguides* which reflects the deficiencies of both the hard-wall tubes and leaky graphs making use of guiding effects of *finite-width potential ditches*.

A magnetic ring chain

Our first example illustrating that a magnetic field is able to *change the spectral properties fundamentally* concerns the *ring chain* discussed in Lecture III; now we suppose that is exposed to a field, in general inhomoheneous, *perpendicular to the graph plane*.



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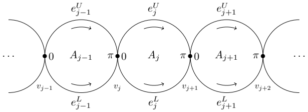

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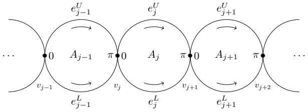

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$$\psi_i(0) = \psi_j(0) =: \psi(0)\,, \quad i,j = 1,\dots,n\,, \quad \sum_{i=1}^{n} \mathcal{D}\psi_i(0) = \alpha\,\psi(0)\,,$$

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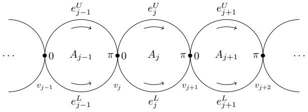

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Here $\alpha \in \mathbb{R}$ is again the coupling constant and we have $n = 4$. In fact, the vector potentials cancel and we get the same condition as before.

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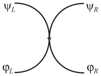
As before the solution comes from analysis of the *basic cell* of the chain,



We use a modified Ansatz $\psi_L(x) = \mathrm{e}^{-iAx}(C_L^+ \mathrm{e}^{ikx} + C_L^- \mathrm{e}^{-ikx})$ for $x \in [-\pi/2, 0]$ and energy $E := k^2 \neq 0$, and similarly for the other three components

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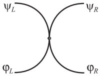

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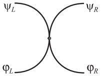

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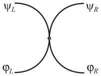

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The functions are again matched through *(a) the δ-coupling* and *(b) Floquet conditions*. Using the function $\eta(k) := 4\cos k\pi + \frac{\alpha}{k}\sin k\pi$, we can write now the spectral condition as

$$|\eta(k)| \leq 4|\cos A\pi|\,;$$

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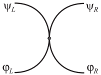

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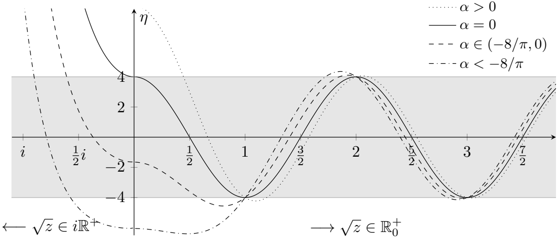
It is illustrative to show the solutions in the graphical form.

In picture: determining the spectral bands

It is easy to see that the degenerate bands referring to *Dirichlet eigenvalues* are not affected to the magnetic field; the other bands can be derived from the picture we have seen before,



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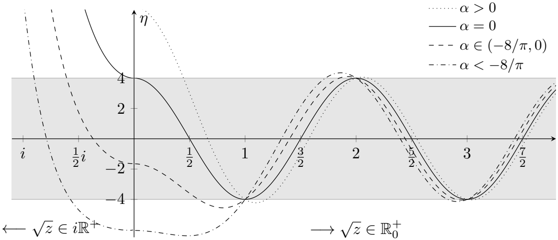

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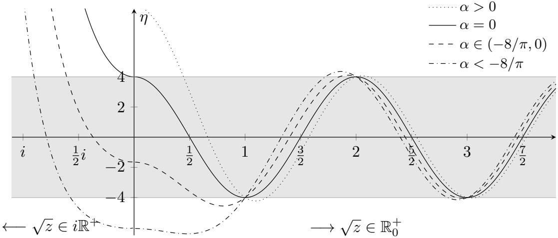

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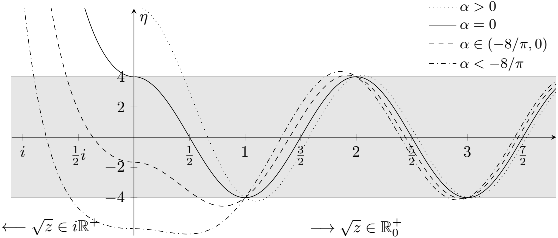

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Conseuently, in the latter case the chain spectrum consists of *infinitely degenerate eigenvalues* only, or *flat bands* as physicists would say, and elementary eigenfunctions are supported by *pairs of adjacent loops*.

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$$\frac{|\cos A_1\pi|}{|\cos A\pi|} > 1,$$

i.e. 'closer the non-magnetic case', otherwise spectrum does not change.

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This simplifies the analysis in the case when the *slope μ is rational*. Indeed, is such a situation we can assume without loss of generality that the sequence $\{A_j\}$ is *periodic* and solve the problem using the Floquet method similarly as we did that for a constant A.

Results of Floquet analysis in the rational case

Theorem

Let $A_j = \mu j + \theta$ *for some* $\mu, \theta \in \mathbb{R}$ *and every* $j \in \mathbb{Z}$. *Then for the spectrum* $\sigma(-\Delta_{\alpha,A})$ *the following holds:*

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(c) *If the situation is as in (b) but $\mu j + \theta + \frac{1}{2} \in \mathbb{Z}$ holds for some $j = 0, \dots, q-1$, then the spectrum $\sigma(-\Delta_{\alpha,A})$ consists of infinitely degenerate eigenvalues only, the Dirichlet ones plus q distinct others in each interval $(-\infty, 1)$ and $\big(n^2, (n+1)^2\big)$.*

P.E., D. Vašata: Cantor spectra of magnetic chain graphs, *J. Phys. A: Math. Theor. 50* (2017), 165201.

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It is particularly simple if the graph in question is *equilateral* like in our example. We consider $\mathfrak{K} := \{k \colon \mathrm{Im}\, k \geq 0 \wedge k \notin \mathbb{Z}\}$ to exclude Dirichlet ev's and seek the spectrum through solution of $(-\Delta_{\alpha,A} - k^2)\begin{pmatrix} \psi(x,k) \\ \varphi(x,k) \end{pmatrix} = 0$

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The idea is to rephrase our *differential operator* problem of the metric graph in term of a *difference equation*, as proposed in the 1980's by physicists, Alexander and de Gennes, followed by mathematicians.

It is particularly simple if the graph in question is *equilateral* like in our example. We consider $\mathfrak{K} := \{k \colon \mathrm{Im}\, k \geq 0 \wedge k \notin \mathbb{Z}\}$ to exclude Dirichlet ev's and seek the spectrum through solution of $(-\Delta_{\alpha,A} - k^2)\binom{\psi(x,k)}{\varphi(x,k)} = 0$

This leads to the difference equation

$$2\cos(A_j\pi)\psi_{j+1}(k) + 2\cos(A_{j-1}\pi)\psi_{j-1}(k) = \eta(k)\psi_j(k)\,, \quad k \in \mathfrak{K}\,,$$

where $\psi_j(k) := \psi(j\pi, k)$ and $\eta(k) := 4\cos k\pi + \frac{\alpha}{k}\sin k\pi$ as above, amended by $\eta(k) = 4 + \alpha\pi$ for $k = 0$.

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What is important, this is a two-way correspondence; we can *reconstruct* the solution of the original problem from that of the difference one.

Duality, continued

Specifically, we have

$$\begin{pmatrix} \psi(x,k) \\ \varphi(x,k) \end{pmatrix} = \mathrm{e}^{\mp iA_j(x-j\pi)}\Big[\psi_j(k)\cos k(x-j\pi)$$

$$+\big(\psi_{j+1}(k)\mathrm{e}^{\pm iA_j\pi} - \psi_j(k)\cos k\pi\big)\frac{\sin k(x-j\pi)}{\sin k\pi}\Big], \quad x \in \big(j\pi,(j+1)\pi\big),$$

and in addition, the function on the left-hand side belongs to $L^p(\Gamma)$ if and only if $\{\psi_j(k)\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ holds with $p \in \{2,\infty\}$.

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This relates weak solutions of the two problems but we can do better:

Theorem

For any interval $J \subset \mathbb{R}\setminus\sigma_D$, the operator $(-\Delta_{\alpha,A})_J$ is unitarily equivalent to the pre-image $\eta^{(-1)}\big((L_A)_{\eta(J)}\big)$, where L_A is the operator on $\ell^2(\mathbb{Z})$ acting as $(L_A\varphi)_j = 2\cos(A_j\pi)\varphi_{j+1} + 2\cos(A_{j-1}\pi)\varphi_{j-1}$.

K. Pankrashkin: Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures, *J. Math. Anal. Appl.* **396** (2012), 640–655.

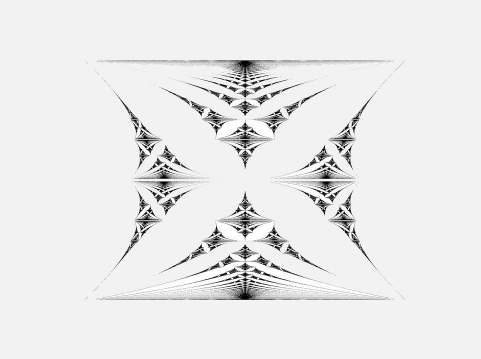
Another way to rephrase the problem

Let me recall the well-known *almost Mathieu equation*

$$u_{n+1} + u_{n-1} + \lambda\cos(2\pi\mu n + \theta))u_n = \epsilon u_n$$

in the *critical case*, $\lambda = 2$, also called *Harper equation*

The spectrum of the corresponding difference operator $H_{\mu,2,\theta}$, independent of θ, as a function of μ is the well-known *Hofstadter butterfly*



Source: Fermat's Library

The Ten Martini Problem

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A. Avila, S. Jitomirskaya: The Ten Martini Problem, *Ann. Math.* **170** (2009), 303–342.

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N.B.: Such a behavior was anticipated in physics half a century ago,

M.Ya. Azbel: Energy spectrum of a conduction electron in a magnetic field, *J. Exp. Theor. Phys.* **19** (1964), 634–645.

and recently confirmed by several groups observing graphene lattices in a homogeneous magnetic field.

How is this related to our problem?

We employ the trick originally proposed in

M.A. Shubin: Discrete magnetic Laplacian, *Commun. Math. Phys. 164* (1994), 259–275.

and consider a *rotation algebra* A_μ generated by elements u, v such that $uv = \mathrm{e}^{2\pi i\mu} vu$. It is simple for $\mu \notin \mathbb{Q}$, thus having *faithful representations*.

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Thus we get a nontrivial result *in a cheap way*: using the duality and the fact that the function η is *locally analytic* we can complete the result from

P.E., D. Vašata: Cantor spectra of magnetic chain graphs, *J. Phys. A: Math. Theor. 50* (2017), 165201.

Theorem

(d) *If $\alpha \neq 0$ and $\mu \notin \mathbb{Q}$, then $\sigma(-\Delta_{\alpha,A})$ does not depend on θ and it is a disjoint union of the isolated-point family $\{n^2 \mid n \in \mathbb{N}\}$ and Cantor sets, one inside each interval $(-\infty, 1)$ and $\left(n^2, (n+1)^2\right)$, $n \in \mathbb{N}$. Moreover, the overall Lebesgue measure of $\sigma(-\Delta_{\alpha,A})$ is zero.*

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Corollary

Let $A_j = \mu j + \theta$ for some $\mu, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. There exists a dense G_δ set of the slopes μ for which, and all θ, the Haussdorff dimension

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Y. Last, M. Shamis: Zero Hausdorff dimension spectrum for the almost Mathieu operator, *Commun. Math. Phys.* **348** (2016), 729–750.

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Corollary

There is another dense set of the slopes μ, *with positive Hausdorff measure, for which, on the contrary,* $\dim_H \sigma(-\Delta_{\alpha,A}) > 0$.

B. Helffer, Qinghui Liu, Yanhui Qu , Qi Zhou: Positive Hausdorff dimensional spectrum for the critical almost Mathieu operator, *Commun. Math. Phys.* **368** (2019), 369–382.

Resonance count

Presence of a magnetic field can influence also other quantum graphs properties. Recall the high-energy asymptotics of the *resonance counting function* we discussed in Lecture III, and expose such a graph Γ with leads to a field described by a vector potential A referring to it as Γ_A.

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Using the technique from [Davies-E-Lipovský'10], reducing the problem to analysis of the core graph with energy-dependent boundary conditions at the 'outer' vertices, one can check the following claim:

If Γ is *Weyl*, $W = \sum_{j=1}^{N} l_j$, then Γ_A is *also Weyl*.

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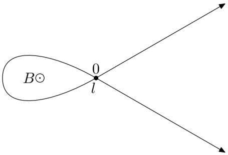
On the other hand: for *non-Weyl* graphs the field may change their effective size. Here is a simple example:



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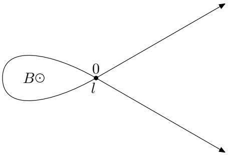

This (Kirchhoff) graph is non-Weyl for $A = 0$, and thus *for any A*.

Resonance count, continued

The resonance condition for such a graph is easily found to be

$$-2\cos\phi + \mathrm{e}^{-ik\ell} = 0,$$

where $\phi = A\ell$ is the magnetic flux through the loop. The senior term, $\mathrm{e}^{ik\ell}$, is missing, so by Langer theorem the effective size is $W = \frac{1}{2}\ell$ provided the *ℓ-independent term is nonzero*.

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Recall that (in the used units) the *flux quantum* is 2π, hence resonances are absent for *odd multiples* of *a quarter* of the quantum. One could compare it with the ring chain where the absolutely continuous spectrum disappeared for *odd multiples* of *one half* of the quantum.

Leaky loops with a magnetic field

Magnetic field effects can also be seen in *leaky graphs*. To give an example, consider a singular interaction supported by a *planar loop* in a homogeneous field with the vector potential $A = \frac{1}{2}B(-x_2, x_1)$.

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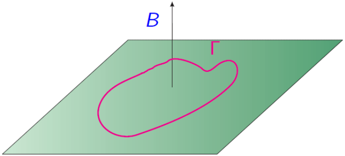
An important physical question concerns the existence of *persistent currents*, in other words, a nonzero probability flux along the loop satisfying the relation

$$\frac{\partial \lambda_n(\phi)}{\partial \phi} = -\frac{1}{c} I_n,$$

where $\lambda_n(\phi)$ is the nth eigenvalue of the Hamiltonian

$$H_{\alpha,\Gamma}(B) := (-i\nabla - A)^2 - \alpha\delta(x - \Gamma)\,;$$

here ϕ is again the magnetic flux (the quantum of which is $2\pi\frac{\hbar c}{|e|}$)



Persistent currents

We can find the strong-coupling asymptotics as we did in Lecture V using the same technique, but a different comparison operator, namely

$$S_\Gamma(B) = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}k(s)^2$$

on $L^2(0,L)$ with $\psi(L-) = \mathrm{e}^{iB|\Omega|}\psi(0+)$ and $\psi'(L-) = \mathrm{e}^{iB|\Omega|}\psi'(0+)$, where Ω is the area encircled by Γ and $B|\Omega|$ is the *flux*.

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Theorem

Let Γ be a C^4-smooth. The for large α the operator $H_{\alpha,\Gamma}(B)$ has a non-empty discrete spectrum and the jth eigenvalue behaves as

$$\lambda_j(\alpha, B) = -\frac{1}{4}\alpha^2 + \mu_j(B) + \mathcal{O}(\alpha^{-1}\ln\alpha),$$

where $\mu_j(B)$ is the jth eigenvalue of $S_\Gamma(B)$ and the error term is uniform in B. In particular, for a fixed j and α large enough the function $\lambda_j(\alpha,\cdot)$ cannot be constant giving rise to a persistent current.

P. Exner, K. Yoshitomi: Persistent currents for 2D Schrödinger operator with a strong δ-interaction on a loop, *J. Phys. A: Math. Gen.* **35** (2002), 3479–3487.

Concentric δ-shells

To give one more example, consider first a specific leaky system the Hamiltonian of which contains δ-interactions supported by *concentric shells*,

$$H_\beta = -\Delta + \beta \sum_n \delta(|x| - r_n) \quad \text{in } L^2(\mathbb{R}^\nu),\ \nu \geq 2$$

with $r_n := \left(n - \frac{1}{2}\right)s,\ d > 0,\ n = 1, 2, \dots,$ and let h_β be the Hamiltonian of the corresponding 1D *Kronig-Penney model*

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The same as is known to be true for *regular, radially periodic* potentials

B.M. Brown, M.S.P. Eastham , A.M. Hinz, T. Kriecherbauer, D.K.R. McCornack, K. Schmidt: Welsh eigenvalues of radially periodic Schrödinger operators, *J. Math. Anal. Appl.* **225** (1998), 347–357.

K.M. Schmidt: Critical coupling constants and eigenvalue asymptotics of perturbed periodic Sturm-Liouville operators, *Commun. Math. Phys.***211** (2000), 645–685.

Welsh eigenvalues vs. Aharonov & Bohm

Let us now insert a *singular magnetic flux* into circles center,

$$H_{\alpha,\beta} = (-i\nabla - A)^2 + \alpha \sum_n \delta(|x| - r_n)\,,$$

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The free Aharonov-Bohm Hamiltonian $H_{\alpha,0} := (-i\nabla - A)^2$ is defined on the *magnetic Sobolev space*. The integer part of α can be removed by a gauge transformation, hence we consider $\alpha \in (0,1)$ only.

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The free Aharonov-Bohm Hamiltonian $H_{\alpha,0} := (-i\nabla - A)^2$ is defined on the *magnetic Sobolev space*. The integer part of α can be removed by a gauge transformation, hence we consider $\alpha \in (0,1)$ only.

The radial symmetry allows us to use the partial wave decomposition. As usual we introduce $U\,:\, L^2(\mathbb{R}_+, r\mathrm{d}r) \to L^2(\mathbb{R}_+)$ acting as $Uf(r) = r^{1/2}f(r)$ to get

$$L^2(\mathbb{R}^2) = \bigoplus_{l\in\mathbb{Z}} U^{-1}L^2(\mathbb{R}_+) \otimes S_l\,,$$

Welsh eigenvalues vs. Aharonov & Bohm, contd.

$$H_{\alpha,0} = \bigoplus_{l} U^{-1} H_{\alpha,0,l} U \otimes I_l \,,$$

where I_l is the identity operator on S_l and the *radial part* is

$$H_{\alpha,0,l} := -\frac{d^2}{d^2 r} + \frac{1}{r^2} c_{\alpha,l} \,, \quad c_{\alpha,l} := -\frac{1}{4} + (l+\alpha)^2,$$

$$D(H_{\alpha,0,l}) := \{ f \in L^2(\mathbb{R}_+) \,:\, -f'' + \frac{c_{\alpha,l}}{r^2} f \in L^2(\mathbb{R}_+),$$
$$\lim_{r\to 0^+} r^{\alpha-1/2} f(r) = 0 \;\text{ if }\; l = 0,$$
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Note that this is a 'pure' Aharonov-Bohm operator without *an additional singular interaction* at the origin

R. Adami, A. Teta: On the Aharonov-Bohm Hamiltonian, *Lett. Math. Phys.* **43** (1998), 43–54.

L. Dąbrowski, P. Šťovíček: Aharonov-Bohm effect with δ type interaction, *J. Math. Phys.* **39** (1998), 47–72.

Welsh eigenvalues vs. Aharonov & Bohm, contd.

Now we add the *δ-interactions* at the points $r = r_n$, $n = 1, 2, \dots,$ and find easily the following elementary properties of $H_{\alpha,\beta}$:

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To this aim one can use *oscillation theory tools* adapting the results of the paper [Schmidt'00, loc.cit.] to our singular interactions.

Welsh eigenvalues vs. Aharonov & Bohm, contd.

The discrete spectrum comes from the partial wave component $H_{\alpha,\beta,0}$ being determined by $c_{\alpha,0} = \alpha^2 - \frac{1}{4}$.

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Let u be the d-periodic real-valued solution of the *1D comparison problem*,

$$h_\beta u = E_\beta u,$$

corresponding to the threshold E_β. Then we make the following claim:

Theorem

Suppose that $\alpha \in (0, \frac{1}{2})$ and put

$$c_{\mathrm{crit}} := -\frac{1}{4}\left(\frac{1}{d}\int_0^d \frac{1}{u^2}\,\mathrm{d}x\right)^{-1}\left(\frac{1}{d}\int_0^d u^2\,\mathrm{d}x\right)^{-1},$$

then E_β is an accumulation point of $\sigma_{\mathrm{disc}}(H_{\alpha,\beta,0})$ provided $\frac{c_{\alpha,0}}{c_{\mathrm{crit}}} > 1$, while for $\frac{c_{\alpha,0}}{c_{\mathrm{crit}}} \leq 1$ the operator has at most finite number of eigenvalues below E_β with the multiplicity taken into account.

P.E., S. Kondej: : Aharonov and Bohm versus Welsh eigenvalues, *Lett. Math. Phys. 108* (2018), 2153–2167.

Welsh eigenvalues vs. Aharonov & Bohm, contd.

Note that in case of regular potentials the number c_{crit} is sometimes called *Knesser constant* in the literature.

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Since $c_{\text{crit}} > -\frac{1}{4}$ by Schwartz inequality, we get

Corollary

There exists an $\alpha_{\text{crit}}(\beta) = \alpha_{\text{crit}} \in (0, \frac{1}{2})$ *such that for* $\alpha \in (0, \alpha_{\text{crit}})$ *the operator* $H_{\alpha,\beta}$ *has infinitely many eigenvalues accumulating at the threshold* E_0*, the multiplicity taken into account, while for* $\alpha \in [\alpha_{\text{crit}}, \frac{1}{2})$ *the cardinality of the discrete spectrum is finite.*

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Moreover, since in our case u is known (quasi)explicitly, we find easily that $\alpha_{\mathrm{crit}}(\beta) = \mathcal{O}(\beta^2)$ holds as $\beta \to 0$ and

$$\alpha_{\mathrm{crit}}(\beta) = \frac{1}{2} + \mathcal{O}(\beta^2 \mathrm{e}^{-|\beta| d/2}) \quad \text{as} \quad \beta \to -\infty\,,$$

$$\alpha_{\mathrm{crit}}(\beta) = \frac{1}{2} + \mathcal{O}(\beta^{-1}) \quad \text{as} \quad \beta \to \infty\,;$$

note that the *sign of β* shows up only in the error term.

Emptiness of $\sigma_{\rm disc}(H_{\alpha,\beta})$ for weak interactions

The next claim comes from properties of the quadratic form of the operator $H_{\alpha,\beta,0} - E_\beta$. Any $f \in D(H_{\alpha,\beta,0})$ can be written as $u\chi$ with $\chi \in H_0^{2,2}(\mathbb{R}_+)$, and

$$q_{\alpha;\beta,0}[u\chi] = -\int_0^\infty u\chi(u\chi)''\,\mathrm{d}r + c_{\alpha,0}\int_0^\infty u^2\frac{\chi^2}{r^2}\,\mathrm{d}r - E_\beta\|u\chi\|^2$$

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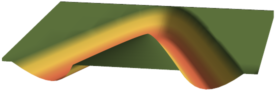
Examining the right-hand side, one can prove that the form is *non-negative for small enough* $|\beta|$, and consequently, referring again to the paper [E-Kondej'18, loc.cit.], we have

Theorem

Given $\alpha \in (0, \frac{1}{2})$, there exists a $\beta_0 > 0$ such that for any $|\beta| < \beta_0$ the operator $H_{\alpha;\beta}$ has empty discrete spectrum.

Soft quantum waveguides

Let us turn to the second topic mentioned in the opening. The leaky wire model with its *zero width* is also an idealization; to get something more realistic we replace the δ function by a finite *potential well*



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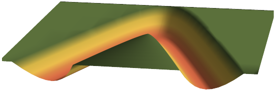

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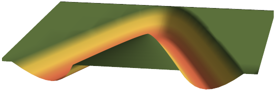

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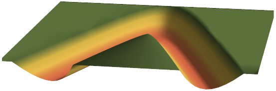

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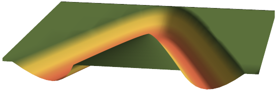

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- (c) $|\Gamma(s) - \Gamma(s')| \to \infty$ holds as $|s - s'| \to \infty$ (no U-shapes, etc.).

The interaction support

Recall that one can *reconstruct the curve* from the knowledge of γ, up to Euclidean transformations: putting $\beta(s_2, s_1) := \int_{s_1}^{s_2} \gamma(s)\,\mathrm{d}s$, we have

$$\Gamma(s) = \left(x_1 + \int_{s_0}^{s} \cos\beta(s_1, s_0)\,\mathrm{d}s_1, x_2 - \int_{s_0}^{s} \sin\beta(s_1, s_0)\,\mathrm{d}s_1\right)$$

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$$\Omega^a := \{x \in \mathbb{R}^2 :\ \mathrm{dist}(x, \Gamma) < a\},$$

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in particular, $\Omega_0^a := \mathbb{R} \times (-a, a)$ corresponds to a straight line for which we use the symbol Γ_0. We assume that

- d Ω^a *does not intersect itself*, in particular, $a\|\gamma\|_\infty < 1$ holds for the strip halfwidth of Γ

which ensures that the points of Ω^a can be uniquely parametrized as follows,

$$x(s, u) = \left(\Gamma_1(s) - u\dot{\Gamma}_2(s), \Gamma_2(s) + u\dot{\Gamma}_1(s)\right),$$

where $N(s) = (-\dot{\Gamma}_2(s), \dot{\Gamma}_1(s))$ is the *unit normal vector* to Γ at the point s.

The potential 'ditch'

We will deal with Schrödinger operators having an *attractive potential* supported in Ω^a. To this aim, we consider

- a nonnegative $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-a, a]$

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It is also useful to introduce the *channel-profile* operator on $L^2(\mathbb{R})$,

$$h_V = -\partial_x^2 - V(x)$$

with the domain $H^2(\mathbb{R})$ which has in accordance with (e) a nonempty and finite discrete spectrum such that

$$\epsilon_0 := \inf \sigma_{\mathrm{disc}}(h_V) = \inf \sigma(h_V) \in \big(-\|V\|_\infty, 0 \big),$$

where the ground-state eigenvalue ϵ_0 is *simple* and the associated eigenfunction $\phi_0 \in H^2(\mathbb{R})$ can be chosen *strictly positive*.

Spectrum of $H_{\Gamma,V}$

If the strip axis Γ is straight, the spectrum is easily found using separation of variables; it is $\sigma(H_{\Gamma_0,V}) = \sigma_{\mathrm{ess}}(H_{\Gamma_0,V}) = [\epsilon_0, \infty)$.

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- a *quantitative criterion* based on Birman-Schwinger principle

Asymptotic results

We know from Lecture IV that $-\Delta - \alpha\delta(x-\Gamma)$ can be approximated in the *norm-resolvent sense* by Schrödinger operators with potentials *transversally scaled*, $V_\varepsilon :\; V_\varepsilon(u) = \frac{1}{\varepsilon} V\left(\frac{u}{\varepsilon}\right)$

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Proposition

Consider a non-straight C^2-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that $|\Gamma(s) - \Gamma(s')| > c|s-s'|$ holds for some $c \in (0,1)$. If the support of its signed curvature γ is noncompact, assume, in addition to (b)*, that $\gamma(s) = \mathcal{O}(|s|^{-\beta})$ with some $\beta > \frac{5}{4}$ as $|s| \to \infty$. Then $\sigma_{\rm disc}(H_{\Gamma,V_\varepsilon}) \neq \emptyset$ holds for all ε small enough.*

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Consider a non-straight C^2-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that $|\Gamma(s) - \Gamma(s')| > c|s-s'|$ holds for some $c \in (0,1)$. If the support of its signed curvature γ is noncompact, assume, in addition to (b), that $\gamma(s) = \mathcal{O}(|s|^{-\beta})$ with some $\beta > \frac{5}{4}$ as $|s| \to \infty$. Then $\sigma_{\mathrm{disc}}(H_{\Gamma,V_\varepsilon}) \neq \emptyset$ holds for all ε small enough.

Consider, on the other hand, a *flat-bottom* waveguide, $V_{J,0}(u) = V_0 \chi_J(u)$, where χ_J refers to an interval $J \subset [-a_0, a_0]$

Asymptotic results

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Consider, on the other hand, a *flat-bottom* waveguide, $V_{J,0}(u) = V_0 \chi_J(u)$, where χ_J refers to an interval $J \subset [-a_0, a_0]$. Using the *high potential wall* limit and the existence result from Lecture I we can conclude:

Proposition

Let Γ be non-straight and assume that assumptions (a)–(d) *are satisfied, then $\sigma_{\mathrm{disc}}(H_{\Gamma,V_\varepsilon}) \neq \emptyset$ holds for all V_0 large enough.*

A quantitative criterion

We have met Birman-Schwinger principle, standard and generalized, in Lecture IV. Since the potential is supported in Ω^a only, we may apply it,

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- use the *curvilinear* (Fermi, parallel) coordinates in Ω^a,
- *'straighten'* the strip and treat $H_{\Gamma,V}$ as a *perturbation* of $H_{\Gamma_0,V}$

Theorem

Let assumptions (a)–(e) be valid and set

$$\mathcal{C}^{\kappa}_{\Gamma,V}(s,u;s',u') = \frac{1}{2\pi}\,\phi_0(u)V(u)\left[(1+u\gamma(s))^{1/2}\,K_0(\kappa|x(s,u)-x(s',u')|)\,(1+u'\gamma(s'))^{1/2}\right.$$
$$\left. -K_0(\kappa|x_0(s,u)-x_0(s',u')|)\,\right]V(u')\phi_0(u')$$

for all (s,u), $(s',u') \in \Omega^a_0$, *then we have* $\sigma_{\mathrm{disc}}(H_{\Gamma,V}) \neq \emptyset$ *provided*

$$\int_{\mathbb{R}^2} \mathrm{d}s\mathrm{d}s' \int_{-a}^{a}\int_{-a}^{a} \mathrm{d}u\mathrm{d}u'\,\mathcal{C}^{\kappa_0}_{\Gamma,V}(s,u;s',u') > 0$$

holds for $\kappa_0 = \sqrt{-\epsilon_0}$.

P.E.: Spectral properties of soft quantum waveguides, *J. Phys. A: Math. Theor.* **53** (2020), 355302.

One more existence result

The integral kernel in the criterion involves the Euclidean distances between points of the curved strip:

$$|x(s,u)-x(s',u')|^2 = |\Gamma(s)-\Gamma(s')|^2 + u^2 + u'^2 - 2uu'\cos\beta(s,s') + 2(u\cos\beta(s,s')-u')\int_{s'}^{s}\sin\beta(\xi,s')\,\mathrm{d}\xi,$$

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where the first term on the right-hand side of this formula, expressing Euclidean distance of points on the strip 'axis', satisfies

$$|\Gamma(s)-\Gamma(s')|^2 = \int_{s'}^{s}\int_{s'}^{s}\cos\beta(\xi,\xi')\,\mathrm{d}\xi\,\mathrm{d}\xi' < |\Gamma_0(s)-\Gamma_0(s')|^2 = |s-s'|^2$$

whenever *the bend is nontrivial*. This property was decisive in the leaky wire case; using it we get from the above theorem the following claim:

Corollary

Let $\mathcal{V}_{\epsilon_0}$ be the family of potentials V satisfying assumptions (d), (e), and $\inf\sigma(h_V)=\epsilon_0$. *Then to any $\epsilon_0>0$ there exists an $a_0=a_0(\epsilon_0)$ such that* $\sigma_{\mathrm{disc}}(H_{\Gamma,V})\neq\emptyset$ *holds for all $V\in\mathcal{V}_{\epsilon_0}$ with* $\mathrm{supp}\,V\subset[-a_0,a_0]$.

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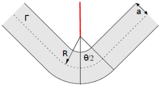

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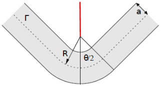

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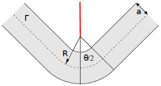

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- Moreover, these results open a plethora of questions about *soft waveguide* properties in different dimensions, different geometries, topological properties of such *potential ditch networks*, etc.
- To quote a fresh result, if you have a family of *soft quantum loops* of a fixed length $|\Gamma|$ and profile V, the *ground state* of the operator $H_{\Gamma,V}$ is *maximized by a circular shape*.

P.E., V. Lotoreichik: Optimization of the lowest eigenvalue of a soft quantum ring, *Lett. Math. Phys.* **111** (2021), 28.

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F. Bentosela, P. Duclos, P. Exner: Absolute continuity in periodic thin tubes and strongly coupled leaky wires, *Lett. Math. Phys.* **65** (2003), 75–82.

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Thank you for your attention!