Constrained quantum dynamics

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With thanks to all my collaborators

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- One may ask general questions, for instance, about the *number of gaps* or about mutual relations between the *band and gap widths*.
- A periodic graphs may be *locally perturbed* which typically gives rise to *localized states*.

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- In both cases the singularity is situated on the *'unphysical sheet'* of energy, that, in an *analytical continuation* of the resolvent/S-matrix.
- In QM, resonances most often come from *perturbations of embedded eigenvalues*; the nontrivial topology of quantum graphs means that they exhibit resonances frequently.

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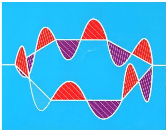

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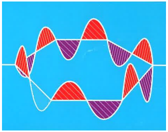

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Let us consider a graph Γ consisting of vertices $\mathcal{V} = \{\mathcal{X}_j : j \in I\}$, finite edges $\mathcal{L} = \{\mathcal{L}_{jn} : (\mathcal{X}_j, \mathcal{X}_n) \in I_{\mathcal{L}} \subset I \times I\}$, and semiinfinite edges (leads) $\mathcal{L}_\infty = \{\mathcal{L}_{j\infty} : \mathcal{X}_j \in I_{\mathcal{C}}\}$. The corresponding state Hilbert space is

$$\mathcal{H} = \bigoplus_{L_j \in \mathcal{L}} L^2([0, l_j]) \oplus \bigoplus_{\mathcal{L}_{j\infty} \in \mathcal{L}_\infty} L^2([0, \infty))\,;$$

its elements we write as columns $\psi = (f_j : \mathcal{L}_j \in \mathcal{L},\, g_j : \mathcal{L}_{j\infty} \in \mathcal{L}_\infty)^{\mathrm{T}}$.

A useful trick

In the absense of external fields, the Hamiltonian acts as $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on each link on $\mathcal{H}^2_{\mathrm{loc}}$ functions satisfying the boundary conditions

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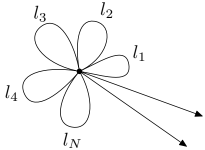

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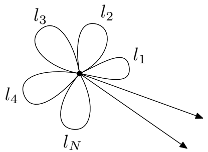

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The coupling in the 'master vertex' is then described by the condition

$$(U - I)\Psi + i(U + I)\Psi' = 0,$$

where the unitary $(2N + M) \times (2N + M)$ matrix U is block-diagonal with the blocks U_j reflecting *the true topology of* Γ.

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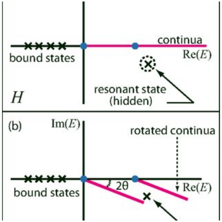

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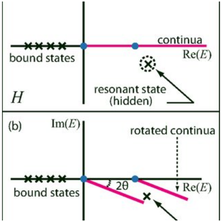

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Quantum graphs we consider are ell suited for application of an *exterior* complex scaling. Looking for complex eigenvalues of the scaled operator we preserve the compact part of the graph using the wave function Ansatz $f_j(x) = a_j \sin kx + b_j \cos kx$ on the j-th internal edge.

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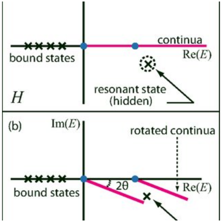

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On the other hand, functions on the semi-infinite edges are scaled by $g_{j\theta}(x) = \mathrm{e}^{\theta/2} g_j(x\mathrm{e}^{\theta})$ with an imaginary θ; the poles of the resolvent on the second sheet become 'uncovered' for *θ large enough*. The 'exterior' boundary values of $g_j(x) = g_j \mathrm{e}^{ikx}$ referring to energy k^2 thus equal to

$$g_j(0) = \mathrm{e}^{-\theta/2} g_j, \quad g_j'(0) = ik\mathrm{e}^{-\theta/2} g_j.$$

Resolvent and scattering resonances

Substituting these boundary values to the matching condition we get

$$\big[(U-I)C_1(k)+ik(U+I)C_2(k)\big]\psi=0,$$

where $\psi=(a_1,b_1,a_2,\dots,b_N,\mathrm{e}^{-\theta/2}g_1,\dots,\mathrm{e}^{-\theta/2}g_M)^{\mathrm{T}}$ and $C_j(k)$ are block- diagonal, $C_j:=\mathrm{diag}\,(C_j^{(1)}(k),C_j^{(2)}(k),\dots,C_j^{(N)}(k),i^{j-1}I_{M\times M})$ with

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Passing to *scattering resonances*, we choose a combination of two planar waves, $g_j=c_j\mathrm{e}^{-ikx}+d_j\mathrm{e}^{ikx}$, as an Ansatz on the external edges; we ask about poles of the matrix $S=S(k)$ which maps the amplitudes of the incoming waves, $c=\{c_n\}$, into the amplitudes of their outgoing counterparts, $d=\{d_n\}$, through the linear relation $d=Sc$.

Resolvent and scattering resonances

Matching the functions at the vertices where the leads are attached, we get

$$(U-I)C_1(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ c_1+d_1 \\ \vdots \\ c_M+d_M \end{pmatrix} + ik(U+I)C_2(k)\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ \vdots \\ b_N \\ d_1-c_1 \\ \vdots \\ d_M-c_M \end{pmatrix} = 0$$

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Proposition

The two above resonance notions, the resolvent and scattering one, are equivalent for quantum graphs.

P.E., J. Lipovský: Resonances from perturbations of quantum graphs with rationally related edges, *J. Phys. A: Math. Theor.* **43** (2010), 105301.

Effective coupling on the compact subgraph

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To this aim, we write U in the block form, $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, where U_1 in the $2N \times 2N$ matric referring to the compact subgraph, U_4 is the $M \times M$ matrix related to the exterior part, and the off-diagonal U_2 and U_3 are rectangular matrices connecting the two.

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Eliminating the external part leads to an effective coupling on the compact subgraph expressed by the condition

$$(\tilde{U}(k) - I)(f_1, \dots, f_{2N})^{\mathrm{T}} + i(\tilde{U}(k) + I)(f_1', \dots, f_{2N}')^{\mathrm{T}} = 0,$$

where the corresponding coupling matrix

$$\tilde{U}(k) := U_1 - (1-k)U_2[(1-k)U_4 - (k+1)I]^{-1}U_3$$

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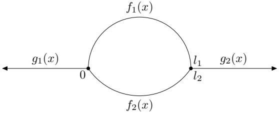
This is another nice illustration of a simple formula know already to *Schur*, often attributed to *Feshbach*, or *Grushin*, or other people.

Example: a loop with two leads



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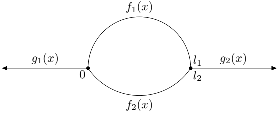

In each vertex we use a four-parameter family of boundary conditions assuming *continuity on the loop*, $f_1(0) = f_2(0)$, together with

$$f_1(0) = \alpha_1^{-1}(f_1'(0) + f_2'(0)) + \gamma_1 g_1'(0),$$
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and similarly in the other vertex with $\alpha_j \in \mathbb{R}$, $\tilde{\alpha}_j \in \mathbb{R}$, and $\gamma_j \in \mathbb{C}$.

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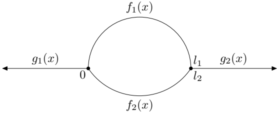

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Writing the loop edge lengths as $l_1 = l(1-\lambda)$ and $l_2 = l(1+\lambda)$ with $\lambda \in [0,1]$, which effectively means shifting one of the connections points around the loop as λ is changing, one arrives at the resonance condition

$$\sin kl(1-\lambda)\sin kl(1+\lambda) - 4k^2\beta_1^{-1}(k)\beta_2^{-1}(k)\sin^2 kl + k[\beta_1^{-1}(k) + \beta_2^{-1}(k)]\sin 2kl = 0,$$

where $\beta_i^{-1}(k) := \alpha_i^{-1} + \frac{ik|\gamma_i|^2}{1-ik\tilde\alpha_i^{-1}}$.

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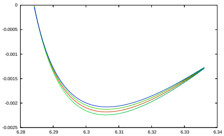

$n = 2$ and $\tilde{\alpha}_1^{-1} = -2$, $\alpha_2^{-1} = 0$, $\alpha_1^{-1} = \tilde{\alpha}_2^{-1} = |\gamma_j|^2 = 1$

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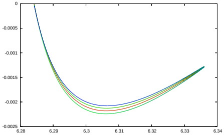

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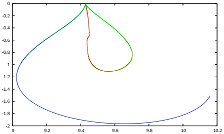

$n = 3$ and all the $\alpha_j^{-1} = \tilde{\alpha}_j^{-1} = |\gamma_j|^2 = 1$

Example: a loop with two leads

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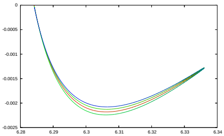

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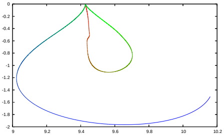

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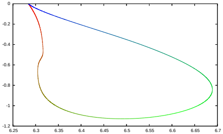
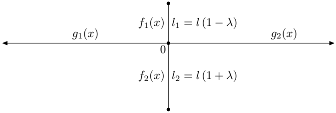


$n = 2$ and the same parameter values

Another example: a cross-shaped graph



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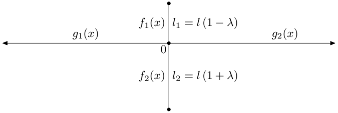

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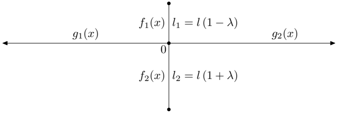

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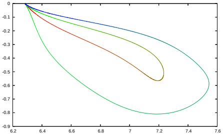

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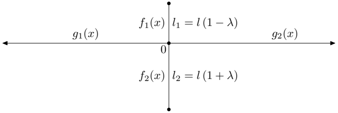

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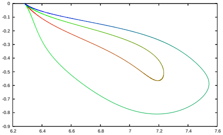

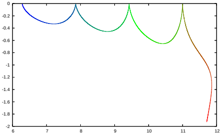

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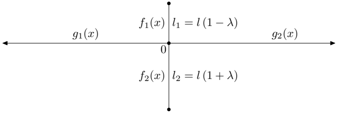

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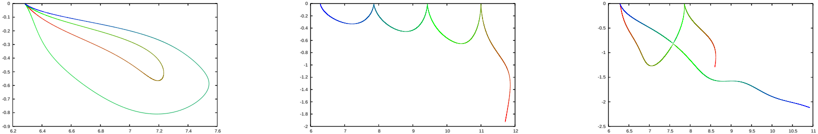

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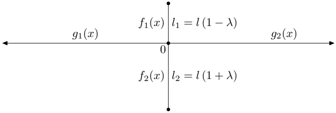

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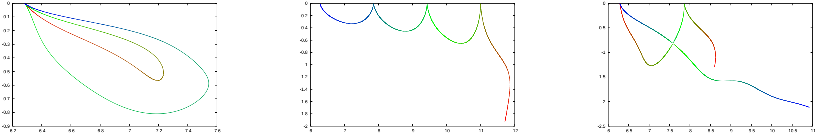

The last one shows an *avoided crossing* of resonance trajectories, the last two also illustrate an effect called *quantum holonomy*.

T. Cheon, A. Tanaka: New anatomy of quantum holonomy, *EPL* **85** (2009), 20001.

High-energy asymptotics

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To understand what is happening it is useful to look at graphs with a general vertex coupling. Denoting $e_j^\pm := \mathrm{e}^{\pm ikl_j}$ and $e^\pm := \prod_{j=1}^N e_j^\pm$, we can write the secular equation determining the singularities is

$$0 = \det\Big\{\frac{1}{2}[(U-I)+k(U+I)]E_1(k)+\frac{1}{2}[(U-I)+k(U+I)]E_2+k(U+I)E_3 \\ +(U-I)E_4+[(U-I)-k(U+I)]\,\mathrm{diag}\,(0,\dots,0,I_{M\times M})\Big\},$$

High-energy asymptotics

where $E_i(k) = \mathrm{diag}\,\left(E_i^{(1)}, E_i^{(2)}, \dots, E_i^{(N)}, 0, \dots, 0\right)$, $i = 1, 2, 3, 4$, consists of a trivial $M \times M$ part and N nontrivial 2×2 blocks

$$E_1^{(j)} = \begin{pmatrix} 0 & 0 \\ -ie_j^+ & e_j^+ \end{pmatrix}, \; E_2^{(j)} = \begin{pmatrix} 0 & 0 \\ ie_j^- & e_j^- \end{pmatrix}, \; E_3^{(j)} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \; E_4^{(j)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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Fortunately, mathematics is eternal; we have an almost century old result:

Theorem

Let $F(k) = \sum_{r=0}^{n} a_r(k)\, \mathrm{e}^{ik\sigma_r}$, where $a_r(k)$ are rational functions of the complex variable k with complex coefficients, and the numbers $\sigma_r \in \mathbb{R}$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_n$. Let us assume that $\lim_{k\to\infty} a_0(k) \neq 0$ and $\lim_{k\to\infty} a_n(k) \neq 0$. Then there are a compact $\Omega \subset \mathbb{C}$, real numbers m_r and positive K_r, $r = 1, \dots, n$, such that the zeros of $F(k)$ outside Ω lie in the logarithmic strips bounded by the curves $-\mathrm{Im}\, k + m_r \log|k| = \pm K_r$ and the counting function of the zeros behaves in the limit $R \to \infty$ as

$$N(R, F) = \frac{\sigma_n - \sigma_0}{\pi} R + \mathcal{O}(1).$$

R.E. Langer: On the zeros of exponential sums and integrals, *Bull. Amer. Math. Soc.* **37** (1931), 213–239.

Application of Langer theorem

Rewriting the secular equation as $F(k) = 0$, we need to find the senior and junior coefficients; by a straightforward computation one can find that $e^{\pm} = \mathrm{e}^{\pm ikV}$, where $V := \sum_{j=1}^{N} l_j$ is the size of the graph core.

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Lemma

$e^{\pm} = \left(\frac{i}{2}\right)^N \det\left[(\tilde{U}(k) - I) \pm k(\tilde{U}(k) + I)\right]$ *with* $\tilde{U}(k)$ *defined above.*

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Given a quantum graph (Γ, H_U) *with finitely many edges and the vertex coupling given by matrices* U_j, *the resonance counting function behaves as*

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where W *is the effective size of* Γ *satisfying* $0 \le W \le V := \sum_{j=1}^{N} l_j$. *Moreover,* $W < V$ *(graph is non-Weyl) if and only there is a vertex such that the matrix* $\tilde{U}_j(k)$ *has an eigenvalue* $(1-k)/(1+k)$ *or* $(1+k)/(1-k)$.

E.B. Davies, P.E., J. Lipovský: Non-Weyl asymptotics for quantum graphs with general coupling conditions, *J. Phys. A: Math. Theor.* **43** (2010), 474013.

Permutation-invariant couplings

Vertex couplings *invariant w.r.t. edge permutations* are described by matrices $U_j = a_j J + b_j I$, where number $a_j,\, b_j \in \mathbb{C}$ such that $|b_j| = 1$ and $|b_j + a_j \deg v_j| = 1$; matrix J has all the entries equal to one. Note that both the δ and δ'_{s} are particular cases of such a coupling.

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For a vertex with p internal and q external edges and such a coupling U_j, the effective matrix matrix $\tilde{U}_j(k)$ is easily calculated; this allows us to make the following conclusion:

Corollary

If (Γ, H_U) has a vertex with a permutation-invariant coupling which is balanced, $p = q$, the graph is non-Weyl if and only if the coupling at this vertex is either of Kirchhoff or anti-Kirchhoff type,

$$f_j = f_n, \;\; \forall j, n \leq 2p, \quad \sum_{j=1}^{2p} f'_j = 0 \quad \text{or} \quad f'_j = f'_n, \;\; \forall j, n \leq 2p, \quad \sum_{j=1}^{2p} f_j = 0$$

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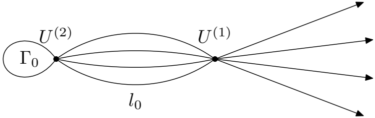
If one drops the requirement of permutation symmetry, it is possible to construct *examples of non-Weyl graphs* in which *no vertex is balanced*.

What is the cause of a non-Weyl asymptotics?

We want to show that (anti-)Kirchhoff conditions at balanced vertices are easy to decouple diminishing thus effectively the graph size.



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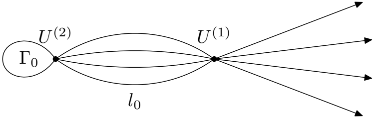

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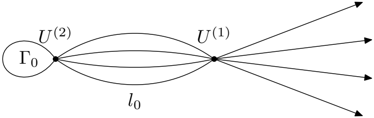

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The idea is to use a *unitary equivalence*. Given a unitary $p \times p$ matrix V we define $V^{(1)} := \mathrm{diag}\,(V, V)$ and $V^{(2)} := \mathrm{diag}\,(I_{(q-p)\times(q-p)}, V)$, then it is straightforward to check that the original graph Hamiltonian is *unitarily equivalent* to the one in which matrices $U^{(1)}$ and $U^{(2)}$ are replaced by $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ and $[V^{(2)}]^{-1}U^{(2)}V^{(2)}$, respectively.

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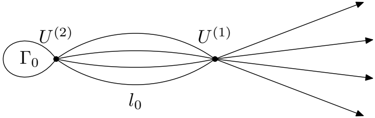

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If the columns of V are *orthonormal eigenvectors* of $U^{(1)}$, beginning with $\frac{1}{\sqrt{p}}\,(1, 1, \dots, 1)^{\mathrm{T}}$, then $[V^{(1)}]^{-1}U^{(1)}V^{(1)}$ decouples then into $2\times$ *blocks*.

What is the cause of a non-Weyl asymptotics?

The first one of those corresponds to the *symmetrization* of all the external u_j's and internal f_j's, thus leading to the 2×2 coupling matrix $U_{2\times 2} = apJ_{2\times 2} + bI_{2\times 2}$; in the complement the internal and external edges are *separated* satisfying Robin conditions, $(b-1)v_j(0) + i(b+1)v_j'(0) = 0$ and $(b-1)g_j(0) + i(b+1)g_j'(0) = 0$ for $j = 2, \dots, p$.

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The 'overall' Kirchhoff/anti-Kirchhoff condition at v_1 is transformed into the *'line' Kirchhoff/anti-Kirchhoff condition* in the subspace of permutation-symmetric functions, and since this is *no coupling at all* (recall that anti-Kirchhhoff and Kirchhoff on line are unitarily equivalent), this causes non-Weyl behavior by effectively *reducing the graph size by l_0*.

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Note that similar trick can used in analysis of *tree graphs* rephrasing the task as an investigation of a family of problems of the line.

A.V. Sobolev, M.Z. Solomyak: Schrödinger operator on homogeneous metric trees: spectrum in gaps, *Rev. Math. Phys.* **14** (2002), 421–467.

Effective size is a global property

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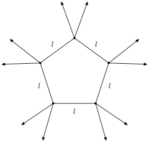
One may ask whether considering the effect of each balanced vertex *separately* allows to to determine the effective size. It is *not* the case, as the following simple example of Kirchhoff graph Γ_n shows:



The symmetry allows to decompose the system w.r.t. the cyclic rotation group $\mathbb{Z}_n$ into segments characterized by numbers ω satisfying $\omega^n = 1$; the resonance condition then reads $-2(\omega^2 + 1) + 4\omega e^{-ik\ell} = 0$

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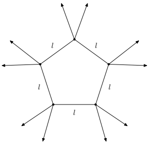

The symmetry allows to decompose the system w.r.t. the cyclic rotation group $\mathbb{Z}_n$ into segments characterized by numbers ω satisfying $\omega^n = 1$; the resonance condition then reads $-2(\omega^2+1) + 4\omega \mathrm{e}^{-ik\ell} = 0$. Using is, we easily find that the effective size of Γ_n is

$$W_n = \begin{cases} n\ell/2 & \text{if } n \neq 0 \pmod 4, \\ (n-2)\ell/2 & \text{if } n = 0 \pmod 4. \end{cases}$$

Note also that one can demonstrate non-Weyl behavior of graph resonances *experimentally* in a model using *microwave networks*:

M. Ławniczak, J. Lipovský, L. Sirko: Non-Weyl microwave graphs, *Phys. Rev. Lett.* **122** (2019), 140503.

Periodic graphs

Let us no pass to graphs which are truly infinite. There is a number of interesting cases here; we restrict our attention to *periodic graphs*, of a great importance if we think of using graphs to model *material structure*.

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$$H = \int_{Q^*} H(\theta)\,\mathrm{d}\theta$$

where the fiber operator $H(\theta)$ acts on $L^2(Q)$, where $Q \subset \mathbb{R}^d$ is *period cell* of the graph and the *quasimomentum* θ runs through the *dual cell* Q^* of the lattice usually called the *Brillouin zone*.

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- has a *band-and-gap structure*

M.Sh. Birman, T.A. Suslina: A periodic magnetic Hamiltonian with a variable metric. The problem of absolute continuity, *St. Petersburg Math. J.* **11** (2000), 203–232.

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For quantum graphs, however, the spectrum of H *is not necessarily absolutely continuous* since they may exhibit *flat bands* coming from the mentioned violation of the unique continuation property. There are also other differences which we will mention below.

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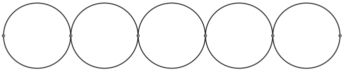

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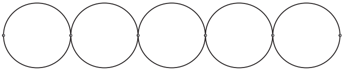

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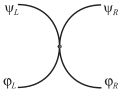
Take the Ansatz $\psi_L(x) = \mathrm{e}^{-iAx}(C_L^+ \mathrm{e}^{ikx} + C_L^- \mathrm{e}^{-ikx})$ for $x \in [-\pi/2, 0]$ and energy $E := k^2 \neq 0$, and similarly for the other three components; for $E < 0$ we put instead $k = i\kappa$ with $\kappa > 0$.



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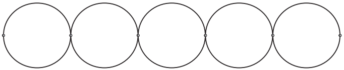

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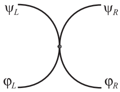

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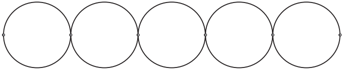

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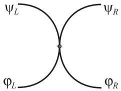

The functions have to be matched through *(a) the δ-coupling* and *(b) Floquet conditions*. This yields equation for the phase factor $\mathrm{e}^{i\theta}$,

$$\sin k\pi \left(\mathrm{e}^{2i\theta} - \tfrac{1}{2}\eta(k)\mathrm{e}^{i\theta} + 1\right) = 0,$$

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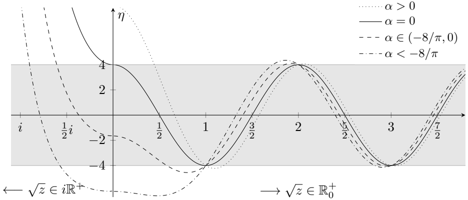

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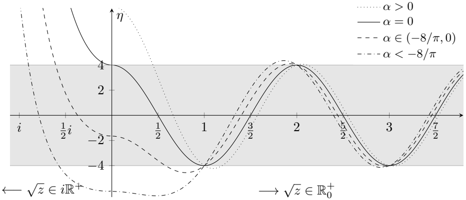

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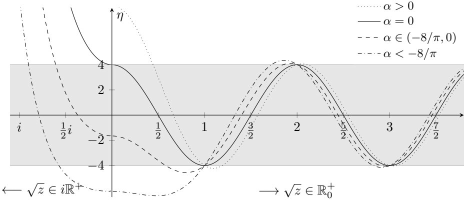

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Note that, up to a factor $\frac{1}{2}$, this nothing but the spectrum of the *Kronig-Penney* model as it is clear from the mirror symmetry of the chain.

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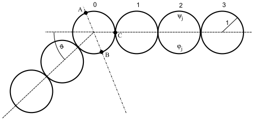

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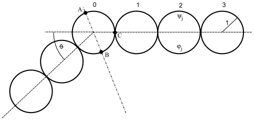

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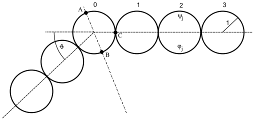

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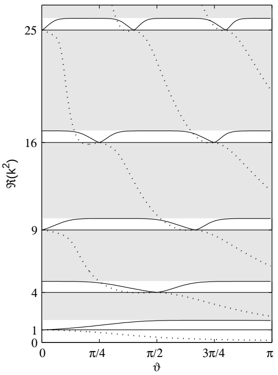
From the general principles we have *at most to eigenvalues* in each gap, because $H_\vartheta^\pm$ and $H_0^\pm$ have a common symmetric restriction with *deficiency indices* $(2,2)$. Furthermore, the *mirror symmetry* allows us to treat the *even* and *odd* parts separately, that is, the halfchain with the Neumann and Dirichlet cut, respectively.

Example: bent-chain spectrum for $\alpha = 3$



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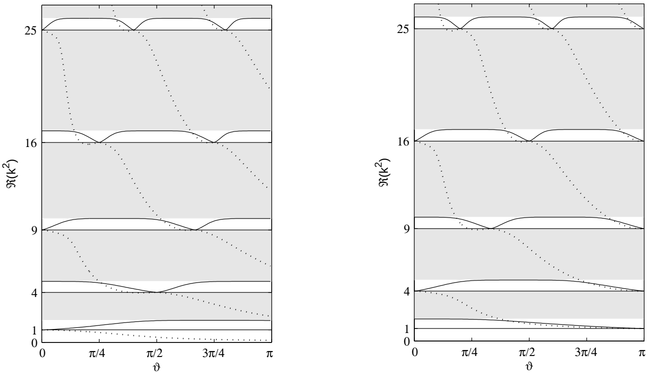

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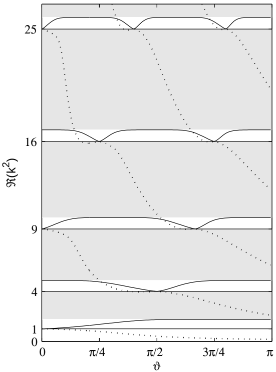

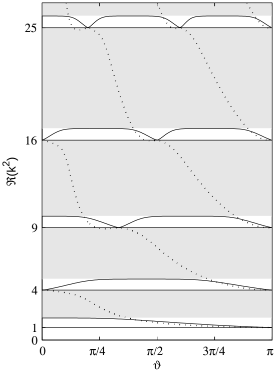

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We see that the eigenvalues in gaps may be absent but only at rational values of ϑ and never simultaneously. Similar pictures we get for other values of α, the dotted lines mark (real values) of *resonance* positions.

P. Duclos, P.E., O. Turek: On the spectrum of a bent chain graph, *J. Phys. A: Math. Theor.* **41** (2008), 415206.

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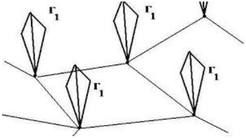

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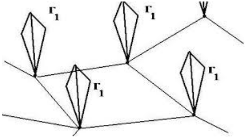

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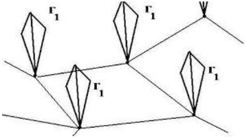

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Thus, instead of 'not a strict law', the question rather is whether *it is a 'law' at all*: do infinite periodic graphs having a *finite nonzero* number of open gaps exist? From obvious reasons we would call them *Bethe-Sommerfeld graphs*.

The answer depends on the vertex coupling

Recall that self-adjointness requires the matching conditions $(U - I)\psi + i(U + I)\psi' = 0$, where ψ, ψ' are vectors of values and derivatives at the vertex of degree n and U is an $n \times n$ unitary matrix

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Worse than that, it was shown that in a 'typical' periodic graph the *probability* of being in a *band* or *gap* is $\neq 0, 1$.

R. Band, G. Berkolaiko: Universality of the momentum band density of periodic networks, *Phys. Rev. Lett.* **113** (2013), 130404.

The existence

Nevertheless, the answer to our question is *affirmative*:

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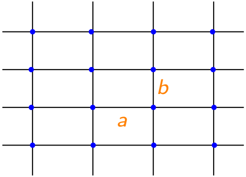
It is sufficient, of course, to demonstrate an example. With this aim we are going to revisit the model of a *rectangular lattice graph* with a δ *coupling* in the vertices introduced in

P.E.: Contact interactions on graph superlattices, *J. Phys. A: Math. Gen.* **29** (1996), 87–102.

P.E., R. Gawlista: Band spectra of rectangular graph superlattices, *Phys. Rev.* **B53** (1996), 7275–7286.



Spectral condition

The Bloch analysis is not difficult in this case. In particular, we find that a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies the *gap condition* which reads

$$2k\left[\tan\left(\frac{ka}{2} - \frac{\pi}{2}\left\lfloor\frac{ka}{\pi}\right\rfloor\right) + \tan\left(\frac{kb}{2} - \frac{\pi}{2}\left\lfloor\frac{kb}{\pi}\right\rfloor\right)\right] < \alpha \quad \text{for } \alpha > 0$$

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we neglect the Kirchhoff case, $\alpha = 0$, which is trivial from the present point of view, $\sigma(H) = [0, \infty)$.

Note that for $\alpha < 0$ the spectrum extends to the negative part of the real axis and may have a gap there – this happens if $\alpha < -4(a^{-1}+b^{-1})$ – which is not important here because there is not more than a single negative gap, and this gap *always extends to positive values*.

What is known about such a quantum graph

The spectrum depends on the ratio $\theta = \frac{a}{b}$. If θ is *rational*, $\sigma(H)$ has clearly *infinitely many gaps* unless $\alpha = 0$ in which case $\sigma(H) = [0, \infty)$

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On the other hand, $\theta \in \mathbb{R}$ is *badly approximable* if there is a $c > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c}{q^2}$$

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Recall that for such numbers one introduces the *Markov constant* by

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(we note that $\mu(\theta) = \mu(\theta^{-1})$) and its *'one-sided analogues'*.

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As an example, take the *golden mean*, $\theta = \frac{\sqrt{5}+1}{2} = [1; 1, 1, \dots]$, which can be regarded as the 'worst' irrational.

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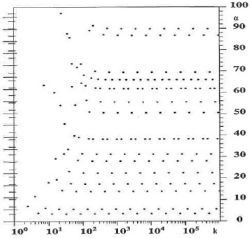

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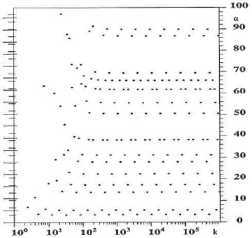

where the points approach the limit values *from above*. Note also that 'higher' gap series open as the coupling strength α increases; the critical values at which that happens are $\frac{\pi^2}{\sqrt{5ab}}\theta^{\pm 1/2}|n^2 - m^2 - nm|,\ n, m \in \mathbb{N}$, cf. [E-Gawlista'96, loc.cit.].

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But a detailed analysis, cf. [E-Turek'17, loc.cit.], shows to a different and more subtle picture:

Theorem

Let $\frac{a}{b} = \theta = \frac{\sqrt{5}+1}{2}$*, then the following claims are valid:*

(i) *If* $\alpha > \frac{\pi^2}{\sqrt{5}a}$ *or* $\alpha \le -\frac{\pi^2}{\sqrt{5}a}$*, there are infinitely many spectral gaps.*

(ii) *If*

$$-\frac{2\pi}{a}\tan\left(\frac{3-\sqrt{5}}{4}\pi\right) \le \alpha \le \frac{\pi^2}{\sqrt{5}a}\,,$$

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Corollary

The above claim about the existence of BS graphs is valid.

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We are also able to control the number of gaps in the BS regime; a more refined Diophantine analysis yields the following result:

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For a given $N \in \mathbb{N}$, there are exactly N gaps in the positive spectrum if and only if α is chosen within the bounds

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Note that the numbers $A_j := \frac{2\pi\left(\theta^{2j} - \theta^{-2j}\right)}{\sqrt{5}} \tan\left(\frac{\pi}{2}\theta^{-2j}\right)$ form an increasing sequence the first element of which is $A_1 = 2\pi \tan\left(\frac{3-\sqrt{5}}{4}\pi\right)$ and

$$A_j < \frac{\pi^2}{\sqrt{5}} \quad \text{holds for all } j \in \mathbb{N}.$$

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Nevertheless, the BS behavior is exceptional and one wonders whether and how often it could be observed in other quantum graph situations.

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