Indicators of many-body quantum chaos and time scales for equilibration

Lea F. Santos

Department of Physics, Yeshiva University, New York, NY, USA

2nd International Summer School on Advanced Quantum Mechanics

STATIC

Goals and Systems

*) Discuss many-body quantum systems

properties (spectrum, eigenstates, observables, dynamics, timescales),

techniques (numerical analysis, support from RMT),

open questions (time for equilibration, dynamical manifestations of chaos, localization, analytical expressions for O(t))

My own questions.

*) Which many-body quantum systems?

Lattice many-body quantum systems

studied with

NMR,

optical lattices,

ion traps.

Far from equilibrium,

time-independent Hamiltonians,

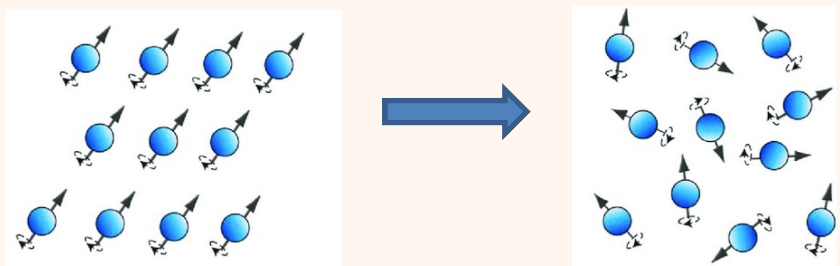
quasi-isolated (coherent evolution).

Coherent Evolution in NMR

NMR

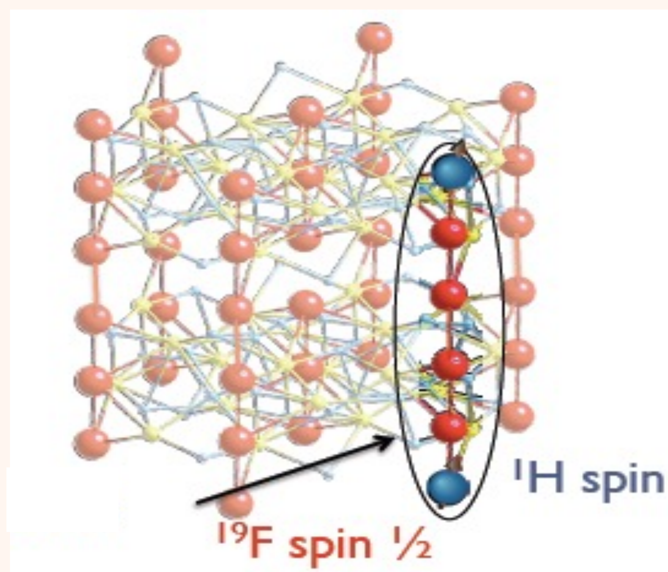
Solid state NMR: nuclear positions are fixed;
They are collectively addressed with magnetic pulses;
Very slow relaxation

Cory (Waterloo)
Cappellaro (MIT)
Ramanathan (Dartmouth)



Fluorine spins-1/2 are arranged in linear chains.

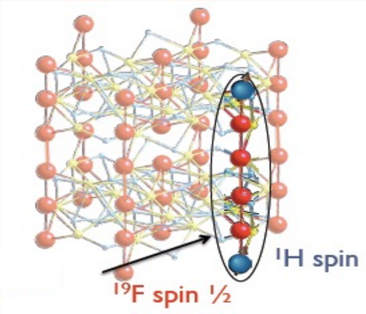
<u>1D Spin-1/2 models</u>



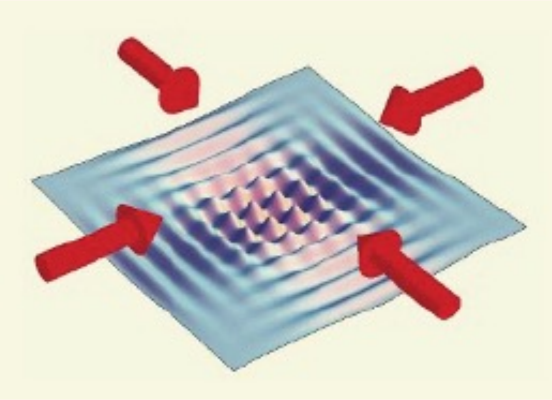
- Coherent evolution
- Pre-thermalization
- Many-body localization

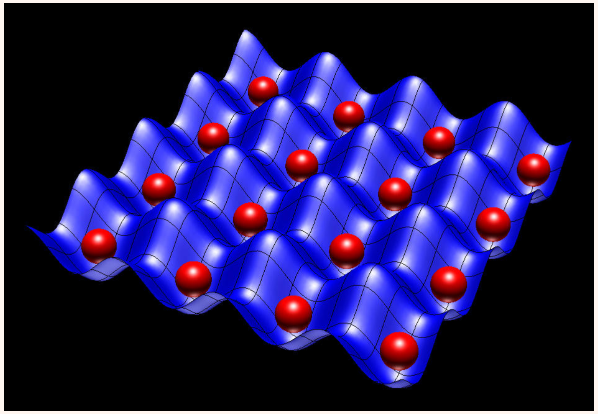
Coherent Evolution in Optical Lattices

NMR



Optical Lattices





Bloch (Max Planck)
Esslinger (ETH)

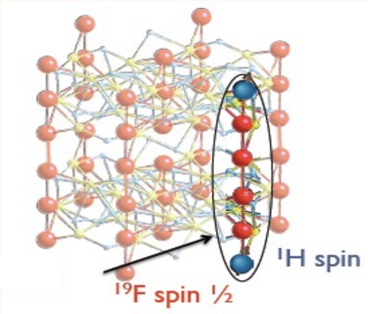
Greiner (Harvard)
Weiss (Penn Sate)

- highly controllable systems – interactions, level of disorder, 1,2,3D
  (simple models)
- quasi-isolated -- study evolution for very long time

Spin models (Heisenberg/Ising)
Bose-Hubbard model

Coherent Evolution in Ion Traps

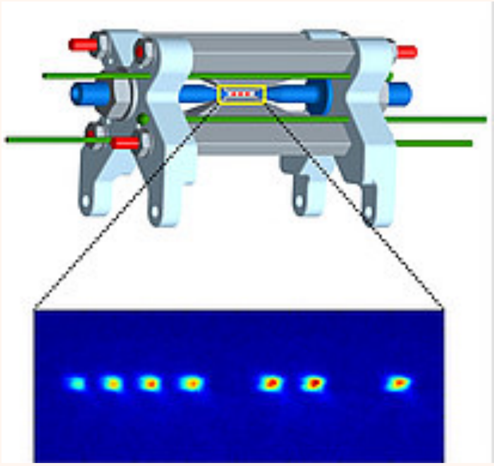
NMR



Ion Traps

Ions trapped via electric and magnetic fields.
Laser used to induce couplings.
Isolated from an external environment.

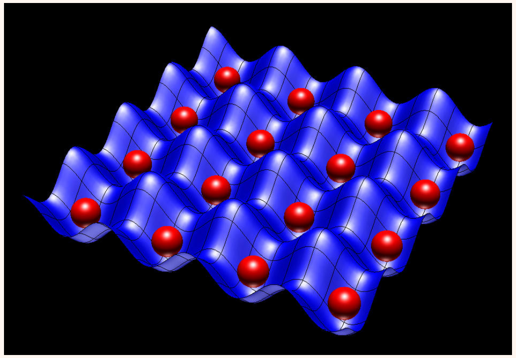
Long coherence times.
Long-range couplings.



Blatt (Innsbrück)

Monroe (Maryland)

Optical Lattices



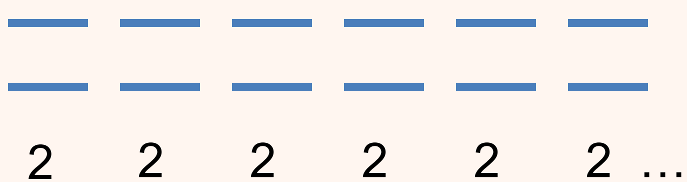
Spin-1/2 models

Obstacles

*) Difficulty for theoretical studies:

-- Few systems can be treated analytically.
Noninteracting, 1D interacting (Bethe ansatz)

-- Numerical methods cannot handle large Hilbert spaces.



Dimension=2^L

2^{10}=1024
2^{20}=1,048,576
(make use of symmetries)

Scaling analysis?
Finite-size effect?

FULL RANDOM MATRICES

Wigner's strategy in the 50's to describe statistically the spectra of heavy nuclei.

Analytical results.

EIGENVALUES OF GOE FULL RANDOM MATRICES

DENSITY OF STATES

LEVEL STATISTICS

Full Random Matrices

Full random matrices from the Gaussian Orthogonal Ensemble (**GOE**):
real and symmetric

Time reversal invariant systems
with rotation symmetry.

- Write a matrix $\boldsymbol{M}$ of dim>10^3
  where all elements are random numbers
  from a Gaussian distribution with mean 0 and variance "ν"
  (ν =1).

Full Random Matrices

Full random matrices from the Gaussian Orthogonal Ensemble (**GOE**): real and symmetric

- Write a matrix $\boldsymbol{M}$ of dim>10^3 where all elements are random numbers from a Gaussian distribution with mean 0 and variance "ν" (ν =1).

- Add this matrix to its transpose to symmetrize it.

$$\boldsymbol{H} = \frac{\boldsymbol{M}+\boldsymbol{M}^T}{2}$$

$$\langle H_{ij} \rangle = 0$$

- The result is a matrix from a GOE.

$$\langle H_{ij}^2 \rangle = \begin{cases} v & i = j \\ v/2 & i \neq j \end{cases}$$

Full Random Matrices: DOS

$$H = \frac{M + M^T}{2}$$

$$\langle H_{ij} \rangle = 0$$

$$\langle H_{ij}^2 \rangle = \begin{cases} v & i = j \\ v/2 & i \neq j \end{cases}$$

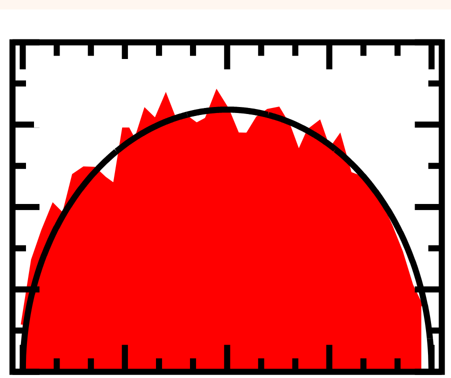
Diagonalize and make a normalized histogram of eigenvalues: **semicircle**

$$\rho(E) = \frac{2}{\pi\,(2\,Dim)}\sqrt{2Dim\,v^2 - E^2}$$

For the limit $Dim \to \infty$

(See Mehta's book for finite *Dim*)

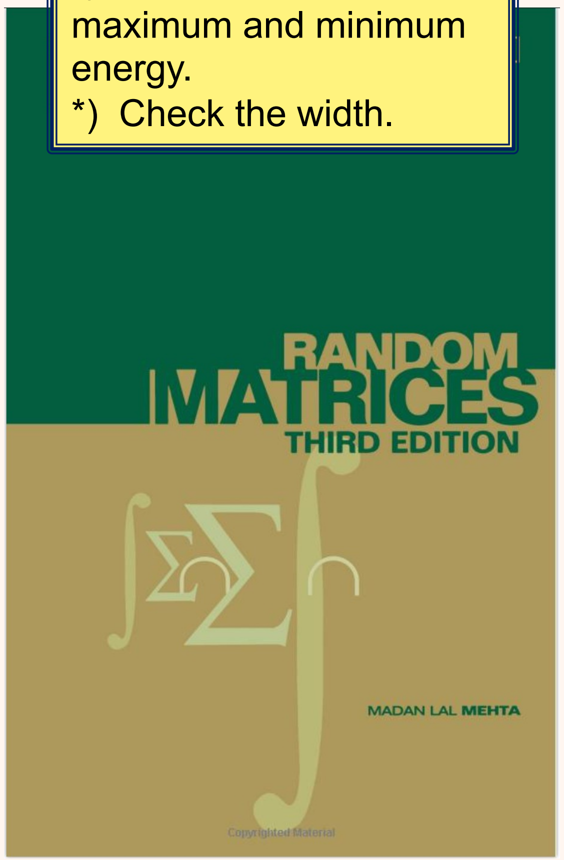
Variance using $\rho(E)$ *is* $\frac{Dim}{2}$



$-\sqrt{2Dim\,v^2}$ E $+\sqrt{2Dim\,v^2}$

Exercise GOE-DOS

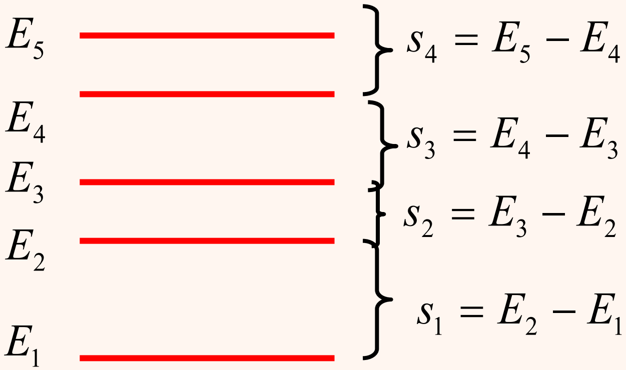
*) Get the DOS for a GOE full random matrix.
*) Confirm the values for maximum and minimum energy.
*) Check the width.



Lea F. Santos, Yeshiva University

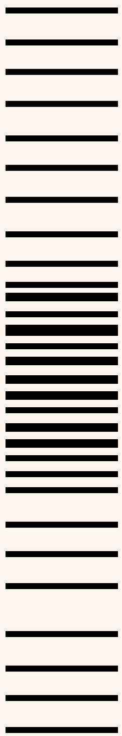
Level Statistics

Level spacing distribution



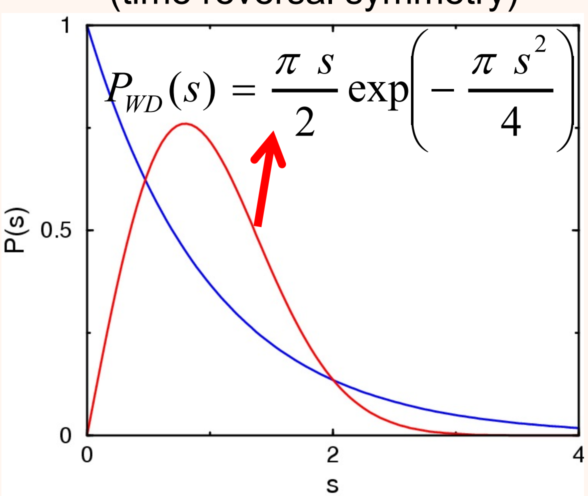
rescale the energies, so that the local density of states of the rescaled eigenvalues is 1

After ***unfolding*** the spectrum



Wigner-Dyson distribution

(time reversal symmetry)



Eigenvalues are correlated
Eigenvalues do not cross
Level repulsion
Rigid spectrum

Guhr et al
Phys. Rep. **299**, 189 (1998)

Level Statistics

Exercise GOE-Ps

Get P(s) for a
GOE full random matrix.

E_5, E_4, E_3, E_2, E_1

$s_4 = E_5 - E_4$

$s_3 = E_4 - E_3$

$s_2 = E_3 - E_2$

$s_1 = E_2 - E_1$

Mean level spacing of that set

$$\bar{s} = \frac{s_4 + s_3 + s_2 + s_1}{4} = \frac{E_5 - E_1}{4}$$

Unfolded spacing (rescaled energies)

$$s_1/\bar{s}, \quad s_2/\bar{s}, \quad s_3/\bar{s}, \quad s_4/\bar{s}$$

Lea F. Santos, Yeshiva University

Quick way to *unfold* the spectrum

rescale the energies, so that the
local density of states of the
rescaled eigenvalues is 1.

(i) Order the spectrum in increasing values of energy.

(ii) Discard some eigenvalues from the edges of the spectrum, where the fluctuations are large.
This is arbitrary. Discard for example 10% of the spectrum.

(iii) Separate the remaining eigenvalues into several small sets of eigenvalues, for example,
sets with 11 eigenvalues each: E_{k+1}, E_{k+2} … E_{k+11}
sets with 10 spacings: $s_1 = E_{k+2} - E_{k+1}$,… $s_{10} = E_{k+11} - E_{k+10}$

(iv) Compute the mean level spacing of the set
$(s_1 + \ldots s_{10})/10 = (E_{k+11} - E_{k+1})/10$

(v) Divide each spacing by the mean level spacing of its particular set.
The mean level spacing of the new set of renormalized energies becomes 1.

Brody distribution

Fit the distribution with the Brody distribution
to get a number that quantifies the proximity to a Wigner-Dyson distribution

$$P(s) = (\beta + 1) b s^{\beta} \exp(-b s^{\beta+1})$$

$$b = \left[\Gamma \left(\frac{\beta + 2}{\beta + 1} \right) \right]^{\beta+1}$$

$\beta \sim 1$ **Wigner-Dyson distribution (chaos)**

$\beta \sim 0$ **Poisson distribution (integrable)**

Different levels of repulsion

$$p_\beta(s) = a_\beta s^\beta \exp\left(-b_\beta s^2\right)$$

$$a_\beta = 2\frac{\Gamma^{\beta+1}((\beta+2)/2)}{\Gamma^{\beta+2}((\beta+1)/2)}$$

$$b_\beta = \frac{\Gamma^2((\beta+2)/2)}{\Gamma^2((\beta+1)/2)}$$

GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$)

$\beta = 1$ Time reversal invariant systems with rotation symmetry

Gaussian Orthogonal Ensemble (GOE)
(real and symmetric)

$$a_1 = \pi/2,\ b_1 = \pi/4$$

$\beta = 2$ Time reversal invariance is violated

Gaussian Unitary Ensemble (GUE)
(hermitian)

$$a_2 = 32/\pi^2,\ b_2 = 4/\pi$$

$\beta = 4$ Time reversal invariant systems with half-integer spin and rotation symmetry

Gaussian Sympletic Ensemble (GSE)
(written in terms of quaternions of Pauli matrices)

$$a_4 = 262144/729\pi^3,\ b_4 = 64/9\pi$$

Izrailev distribution

$$p_\beta(s) = a_\beta s^\beta \exp\left(-b_\beta s^2\right)$$

$$a_\beta = 2\frac{\Gamma^{\beta+1}((\beta+2)/2)}{\Gamma^{\beta+2}((\beta+1)/2)}$$

$$b_\beta = \frac{\Gamma^2((\beta+2)/2)}{\Gamma^2((\beta+1)/2)}$$

GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$)

Fit the distribution with the Izrailev's distribution
to get a number that quantifies the proximity to a Wigner-Dyson distribution

$$P_\beta(s) = A\left(\frac{\pi s}{2}\right)^\beta \exp\left[-\frac{1}{4}\beta\left(\frac{\pi s}{2}\right)^2 - \left(Bs - \frac{\beta}{4}\pi s\right)\right]$$

The parameters A and B are obtained from the normalization conditions:

$$\int_0^\infty P_\beta(s)ds = 1 \text{ and } \int_0^\infty sP_\beta(s)ds = 1$$

Izrailev
Phys. Rep. **196**, 299 (1990)

Ratio of consecutive level spacings

$$r_n = \frac{s_n}{s_{n-1}} = \frac{(E_n - E_{n-1})}{(E_{n-1} - E_{n-2})}$$

Equivalent to the GOE Wigner-Dyson: $$P_{WD}(r) = \frac{27}{8} \frac{(r + r^2)}{(1 + r + r^2)^{5/2}}$$

Equivalent to the Poisson: $$P_P(r) = \frac{1}{(1 + r)^2}$$

Exercise GOE-$\langle \tilde{r} \rangle$
Compute $\langle \tilde{r} \rangle$ for a GOE full random matrix.

A number that quantifies the proximity to a Wigner-Dyson distribution

$$\tilde{r}_n = \min \left(\frac{s_n}{s_{n-1}}, \frac{s_{n-1}}{s_n} \right)$$

$\langle \tilde{r} \rangle = 4 - 2 \ln 3 \approx 0.54$ **Wigner-Dyson (GOE)**

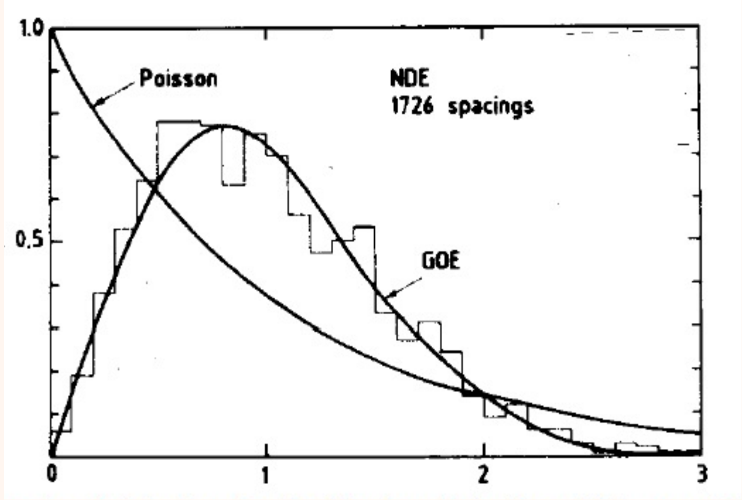
$\langle \tilde{r} \rangle = 2 \ln 2 - 1 \approx 0.39$ **Poisson distribution**

Lea F. Santos, Yeshiva University

PRL **110**, 084101 (2013)

Level Spacing Distribution in Nuclei

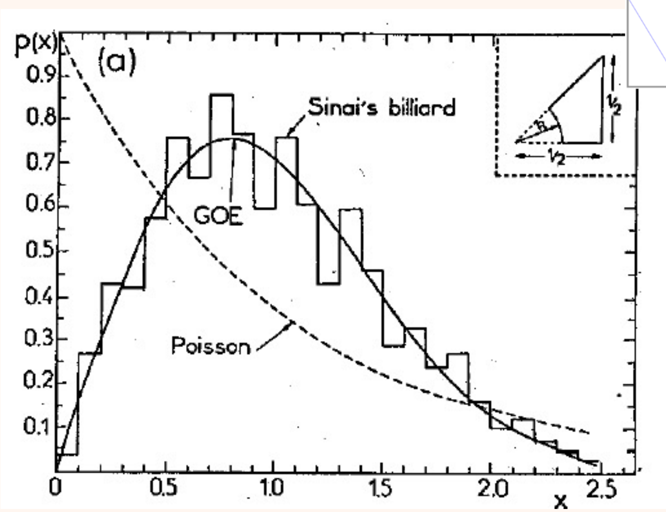
Nearest neighbor spacing distribution for the "**Nuclear Data Ensemble**" comprising 1726 spacings s = S/D with D the mean level spacing and S the actual spacing.

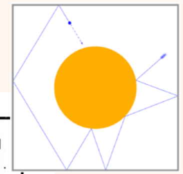


Bohigas, Haq and Pandey (1983)
Nuclear Data for Science and Technology

Level Spacing Distribution and Chaos

The nearest neighbor spacing distribution versus s for the ***quantum*** **Sinai billiard**. The histogram comprises about 1000 consecutive eigenvalues.





Level Statistics vs Classical chaos

Quantum chaos
=
signatures of classical chaos found in the quantum domain

Correspondence well established for systems with
few degrees of freedom

*) G. Casati, F. Valz-Gris, and I. Guarneri, On the connection between quantization of nonintegrable systems and statistical theory of spectra,
Lett. Nuovo Cimento 28, 279 (1980).
*) O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws,
Phys. Rev. Lett. 52, 1 (1984).

Short vs Long-Range Correlations

- Level spacing distribution P(s)
- Ratio of consecutive level spacings P(r)

} detect SHORT-range correlations

A more complete analysis of the spectrum calls for quantities that detect long-range correlations also.

- Level Number Variance

$$\Sigma^2(l) \equiv \langle [N(l,\epsilon)]^2 \rangle - \langle N(l,\epsilon) \rangle^2$$

(i) Discard edges and **unfold** the spectrum.
(ii) Partition the remaining eigenvalues into intervals of length l.
(iii) Count the number of levels inside each interval and compute the variance of the distribution of these numbers.
(iv) Repeat this procedure for various l's.

Lea F. Santos, Yeshiva University

Short vs Long-Range Correlations

- Level spacing distribution P(s)
- Ratio of consecutive level spacings P(r)

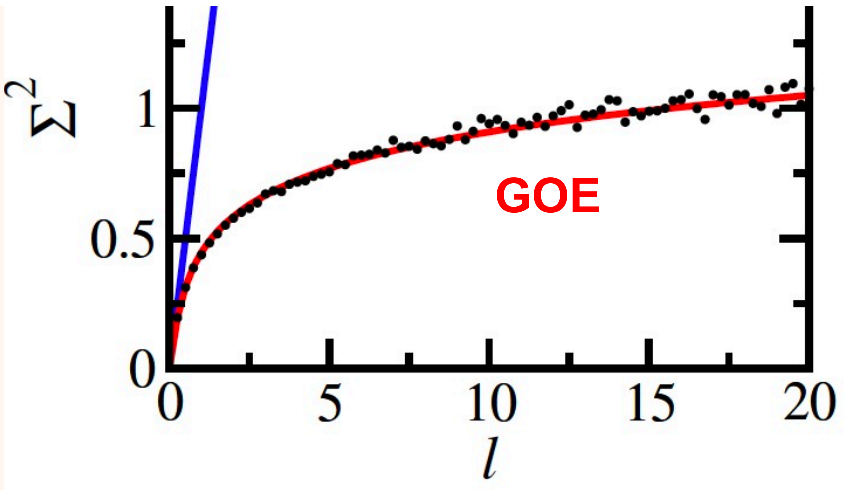
} detect SHORT-range correlations

A more complete analysis of the spectrum calls for quantities that detect long-range correlations also.

$$\Sigma^2(\ell) = \frac{2}{\pi^2}\left(\ln(2\pi\ell) + \gamma_e + 1 - \frac{\pi^2}{8}\right)$$

- Spectral rigidity
- Level Number Variance

$$\Sigma^2(l) \equiv \langle [N(l,\epsilon)]^2 \rangle - \langle N(l,\epsilon) \rangle^2$$



Exercise GOE-LNV

Compute the level number variance for a GOE full random matrix

Unfolding Procedure

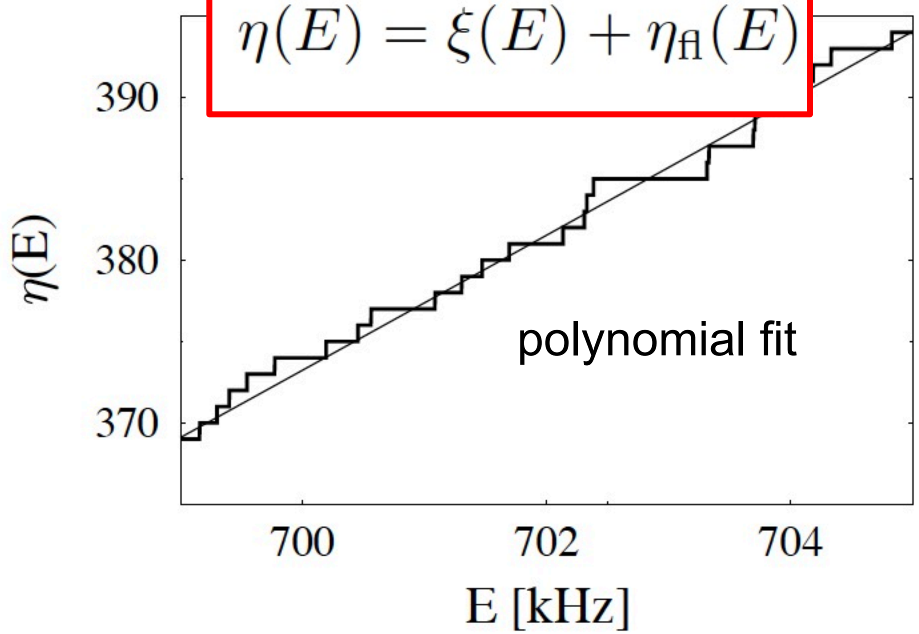
DOS

$$S(E) = \sum_{n=1}^{N} \delta(E - E_n)$$

$$\eta(E) = \int_{-\infty}^{E} S(E')dE'$$

This function counts the number of levels with energy less than or equal to E and is also referred to as the staircase function.

$$\eta(E) = \xi(E) + \eta_{\text{fl}}(E)$$



Instead of the original sequence

$$\{E_1, E_2, \ldots, E_N\}$$

We use

$$\{\xi_1, \xi_2, \ldots, \xi_N\}$$

where

$$\xi_n = \xi(E_n)$$

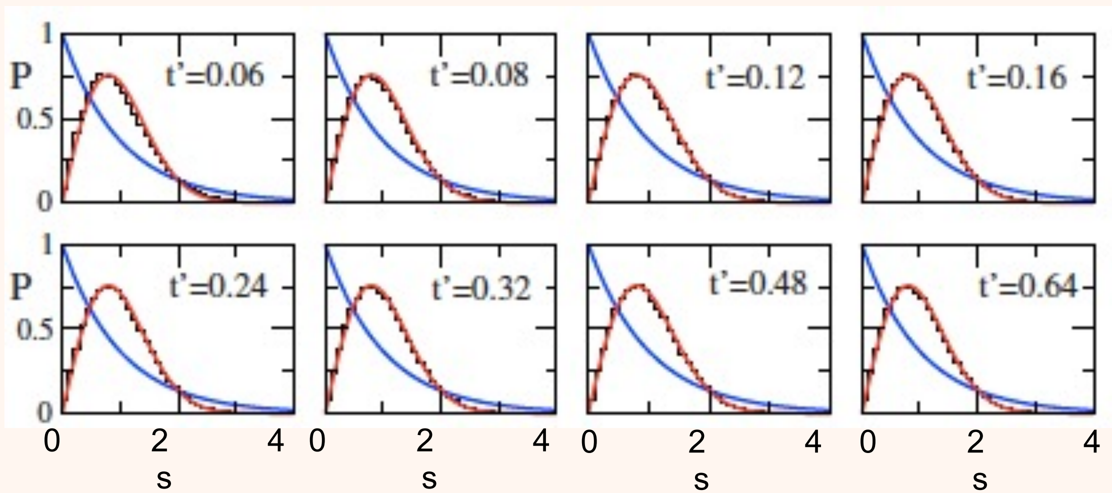
Guhr et al
Phys. Rep. **299**, 189 (1998)

The mean level density of the unfolded spectrum is 1

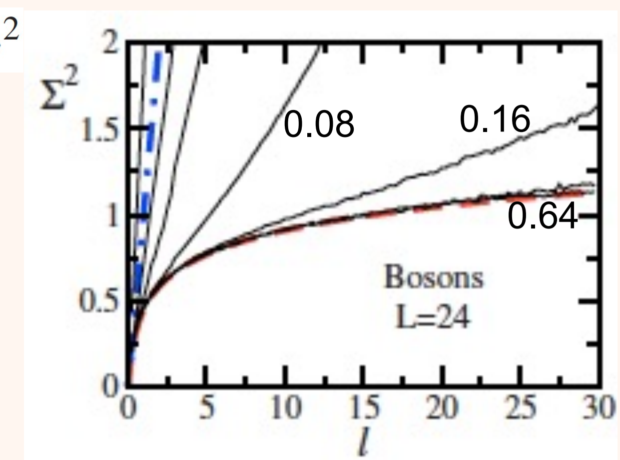
Short vs Long-Range Correlations

$$H = \sum_{n=1}^{L}\left[t\left(b_n^{+}b_n - \frac{1}{2}\right)\left(b_{n+1}^{+}b_{n+1} - \frac{1}{2}\right) - t\left(b_n^{+}b_{n+1} + h.c.\right)\right] + \sum_{n=1}^{L}\left[t'\left(b_n^{+}b_n - \frac{1}{2}\right)\left(b_{n+2}^{+}b_{n+2} - \frac{1}{2}\right) - t'\left(b_n^{+}b_{n+2} + h.c.\right)\right]$$

Hardcore bosons



$$\Sigma^2(l) \equiv \langle [N(l,\epsilon)]^2 \rangle - \langle N(l,\epsilon) \rangle^2$$



GOE

LFS & Rigol
PRE **80** 036206 (2010)

EIGENSTATES OF
GOE FULL RANDOM MATRICES

PORTER-THOMAS DISTRIBUTION

RÉNYI ENTROPY, PARTICIPATION RATIO
ENTANGLEMENT ENTROPY

Porter-Thomas Distribution

Eigenstates of GOE full random matrices are random vectors:
(normalized)

$$|\alpha\rangle = \sum_{n=1}^{D} C_n^\alpha |n\rangle \qquad |\alpha\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \ldots \end{pmatrix}$$

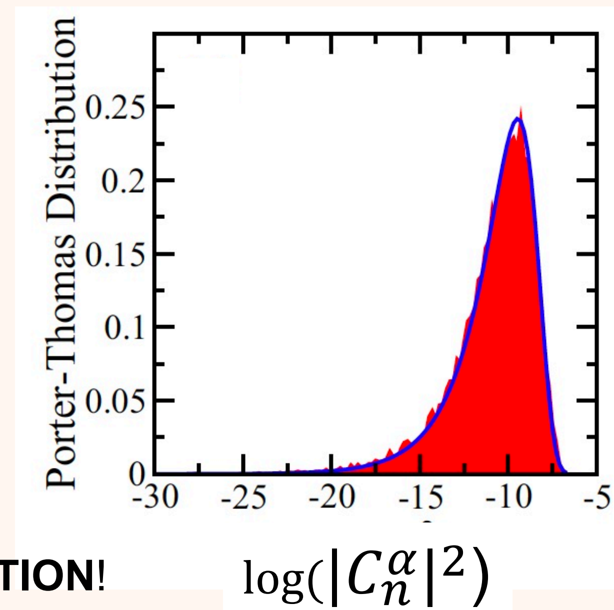
Components are REAL random numbers from a Gaussian distribution.

$$P_{Gaussian}(C_n^\alpha) = \sqrt{\frac{Dim}{2\pi}} \exp\left(-\frac{Dim\,(C_n^\alpha)^2}{2}\right)$$

The weights $(C_n^\alpha)^2$ follow the **Porter-Thomas distribution**

$$PT((C_n^\alpha)^2) = \sqrt{\frac{Dim}{2\pi\,(C_n^\alpha)^2}} \exp\left(-\frac{Dim(C_n^\alpha)^2}{2}\right)$$

The chi-square distribution of degree 1



Take into account **ORTHONORMALIZATION**!

Eigenstates: Participation Ratio

Eigenstates of full random matrices are random vectors: $|\alpha\rangle = \sum_{n=1}^{D} C_n^{\alpha} |n\rangle$

$$|\alpha\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \cdots \end{pmatrix}$$

Components are random numbers from a Gaussian distribution.

Participation Ratio

$$PR^{(\alpha)} \equiv \frac{1}{\sum_{n=1}^{D} |C_n^{(\alpha)}|^4}$$

PR: small (**localization**)

PR=1 (eigenstate = basis)
(extreme localization)

PR: large (**delocalization**)

PR=dim

$|C_n| = \sqrt{\frac{1}{dim}}$

(basis is not well defined in full random matrices)

Eigenstates: Participation Ratio

Eigenstates of full random matrices are random vectors:
Components are random numbers from a Gaussian distribution

$$|\alpha\rangle = \sum_{n=1}^{D} C_n^{\alpha} |n\rangle$$

Normalization: $\sum_{n=1}^{Dim} |C_n^{\alpha}|^2 = 1$

$$\mathcal{P}(C) = \sqrt{D/(2\pi)} e^{-DC^2/2}$$

$$\langle C \rangle = 0 \qquad \langle C^2 \rangle = 1/D$$

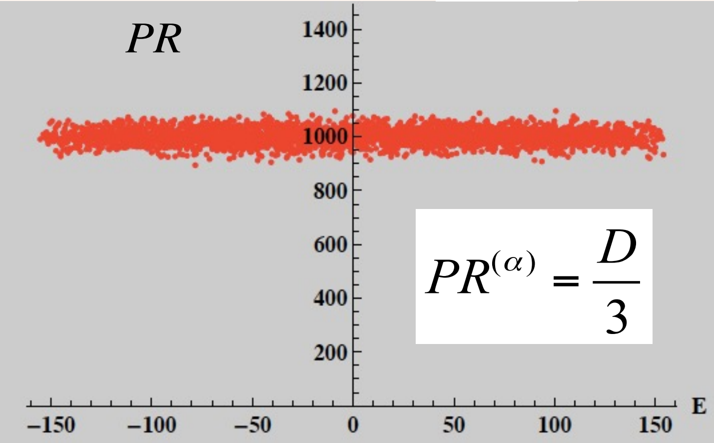
$$\langle C^4 \rangle = 3/D^2$$

$$PR^{(\alpha)} \equiv \frac{1}{\sum_{n=1}^{D} |C_n^{(\alpha)}|^4} \sim \frac{\boldsymbol{D}}{\boldsymbol{3}}$$

$$= \frac{\boldsymbol{D+2}}{\boldsymbol{3}}$$

(GOE)

$$\frac{1}{\sum_{n=1}^{D} |C_n^{\alpha}|^4} = \frac{D}{D\sum_{n=1}^{D} |C_n^{\alpha}|^4} = \frac{D^2}{D * 3} = \frac{\boldsymbol{D}}{\boldsymbol{3}}$$



$$PR^{(\alpha)} = \frac{D}{3}$$

Lea F. Santos, Yeshiva University

Eigenstates: Entropies

(Participation) Rényi entropies: $S_\lambda = \frac{1}{1-\lambda} log\left(\sum_{n=1}^{D} |C_n^\alpha|^{2\lambda}\right)$

- 1st order Rényi entropy = Shannon entropy:

$$S_1 = -\sum_{n=1}^{D} |C_n^\alpha|^2 log(|C_n^\alpha|^2) \qquad S^{GOE}{}_1 = \log(0.48\, dim)$$

- 2nd order Rényi entropy:

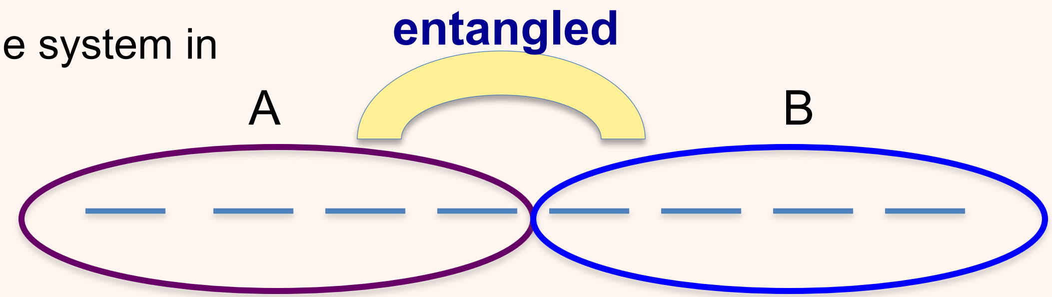
$$S_2 = -log\left(\sum_{n=1}^{D} |C_n^\alpha|^4\right) \qquad S^{GOE}{}_2 = \log(dim) \text{ - } \log(3)$$

Eigenstates: Entanglement Entropy

Entanglement entropy is used to quantify the amount of entanglement in a state.

Its computation requires the bipartition of the system in subsystems A and B and
the partial trace of one of the two.

$$\rho_A = \mathrm{Tr}_B \rho$$



The entanglement entropy is the von Neumann entropy of the reduced density matrix

$$S_{vN} = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right)$$

For a pure random state of dimension $\mathcal{D}=\mathcal{D}_{\mathcal{A}}.\mathcal{D}_{\mathcal{B}}$

$$S_{vN}^{rand} \simeq \ln \mathcal{D}_A - \mathcal{D}_A/(2\mathcal{D}_B)$$

Page
PRL **71**, 1291 (1993)

Eigenstates

Eigenstates of full random matrices are random vectors:
Components are random numbers from a Gaussian distribution

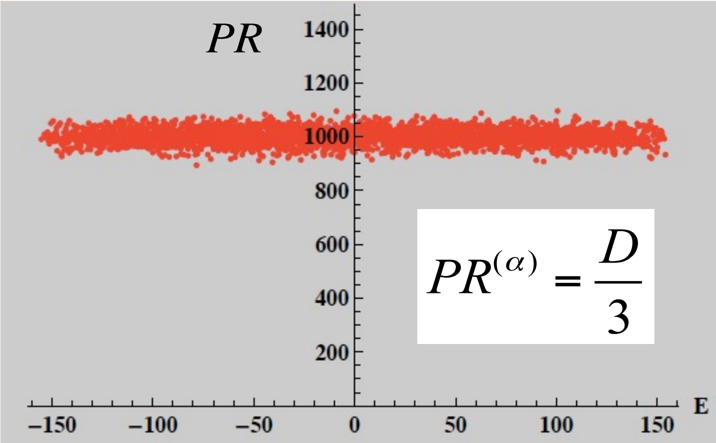
$$|\alpha\rangle = \sum_{n=1}^{D} C_n^{\alpha} |n\rangle$$

Exercise GOE-Psi

Find the eigenstates of a GOE full random matrix and compute:

*) Distribution of the coefficients
*) Porter-Thomas distribution
*) Rényi entropies
*) Participation Ratio
*) Entanglement entropy

$$PR^{(\alpha)} \equiv \frac{1}{\sum_{n=1}^{D} |C_n^{(\alpha)}|^4}$$



$$PR^{(\alpha)} = \frac{D}{3}$$

Lea F. Santos, Yeshiva University

BANDED RANDOM MATRICES

Hamiltonian matrix: Spin-1/2 model NN couplings

$$H = \sum_{n=1}^{L} \frac{h_n}{2}\sigma_n^z + \sum_{n=1}^{L} \frac{J}{4}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z)$$

	110000	101000	100100	100010	100001	011000	010100	010010	010001	001100	001010	001001	000110	000101	000011
110000	H_{11}	J/2	0	0	0	0	0	0	0	0	0	0	0	0	0
101000	J/2	H_{22}	J/2	0	0	J/2	0	0	0	0	0	0	0	0	0
100100	0	J/2	H_{33}	J/2	0	0	J/2	0	0	0	0	0	0	0	0
100010	0	0	J/2	H_{44}	J/2	0	0	J/2	0	0	0	0	0	0	0
100001	0	0	0	J/2	H_{55}	0	0	0	J/2	0	0	0	0	0	0
011000	0	J/2	0	0	0	H_{66}	J/2	0	0	0	0	0	0	0	0
010100	0	0	J/2	0	0	J/2	H_{77}	J/2	0	J/2	0	0	0	0	0
010010	0	0	0	J/2	0	0	J/2	H_{88}	J/2	0	J/2	0	0	0	0
010001	0	0	0	0	J/2	0	0	J/2	H_{99}	0	0	J/2	0	0	0
001100	0	0	0	0	0	0	J/2	0	0	H_{1010}	J/2	0	0	0	0
001010	0	0	0	0	0	0	0	J/2	0	J/2	H_{1111}	J/2	J/2	0	0
001001	0	0	0	0	0	0	0	0	J/2	0	J/2	H_{1212}	0	J/2	0
000110	0	0	0	0	0	0	0	0	0	0	J/2	0	H_{1313}	J/2	0
000101	0	0	0	0	0	0	0	0	0	0	0	J/2	J/2	H_{1414}	J/2
000011	0	0	0	0	0	0	0	0	0	0	0	0	0	J/2	H_{1515}

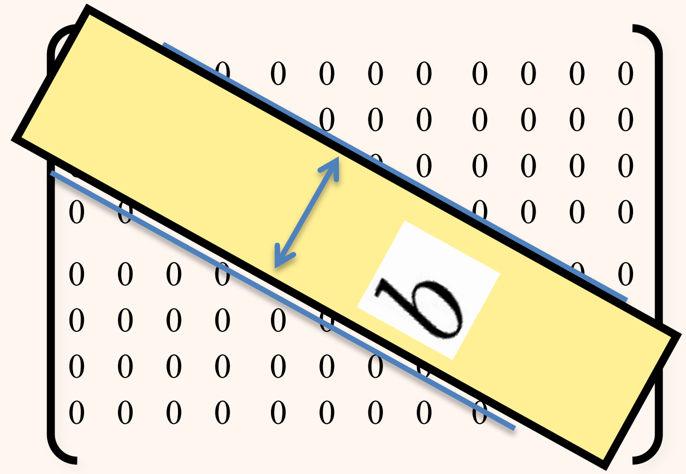
Wigner Band Random Matrix

FULL RANDOM MATRIX:
<u>Hamiltonians are real and symmetric</u>
Gaussian Orthogonal Ensemble (GOE)

0.23	-0.09	1.13	-0.22	0.59	0.58	-0.46	-0.43	-0.46	-1.12	0.90	-0.92
-0.09	-0.02	-0.04	-0.58	0.65	0.05	-0.20	-0.14	0.06	-0.50	1.29	-0.42
1.13	-0.04	0.17	0.55	1.31	0.36	-0.24	0.05	0.49	0.65	-1.18	-0.40
-0.22	-0.58	0.55	0.79	-0.20	-0.03	-0.68	0.16	1.58	0.15	-0.56	0.15
0.59	0.65	1.31	-0.20	-0.79	-0.19	-1.15	0.59	1.14	1.21	-0.25	0.92
0.58	0.05	0.36	-0.03	-0.19	0.59	1.46	0.96	-0.66	0.05	-0.30	0.88
-0.46	-0.20	-0.24	-0.68	-1.15	1.46	-0.80	0.61	0.07	0.15	-0.11	0.28
-0.43	-0.14	0.05	0.16	0.59	0.96	0.61	0.68	-0.59	-0.40	-0.47	-0.08
-0.46	0.06	0.49	1.58	1.14	-0.66	0.07	-0.59	0.82	-0.31	-0.08	0.42
-1.12	-0.50	0.65	0.15	1.21	0.05	0.15	-0.40	-0.31	0.02	-0.95	0.58
0.90	1.29	-1.18	-0.56	-0.25	-0.30	-0.11	-0.47	-0.08	-0.95	-0.41	0.03
-0.92	-0.42	-0.40	0.15	0.92	0.88	0.28	-0.08	0.42	0.58	0.03	-0.36

Basis is ill defined

Wigner Band Random Matrix:
<u>Hamiltonians are real and symmetric</u>



Diagonal elements: integers

$$\cdots, -2, -1, 0, 1, 2, \cdots$$

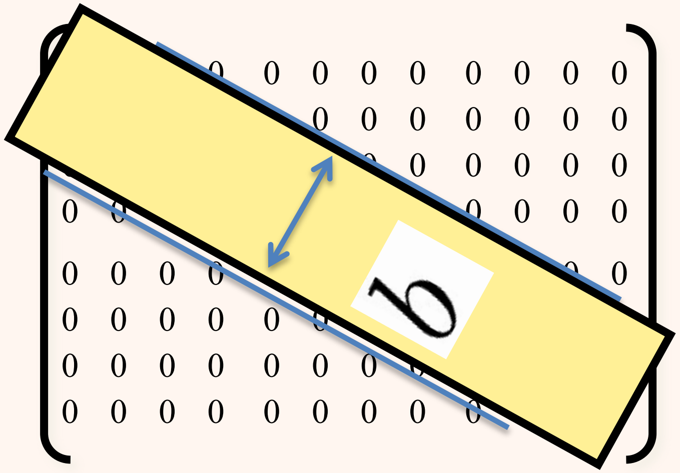
Off-diagonal elements $|v_{mn}| = v$

the signs of the v_{ij} are random

ANNALS OF MATHEMATICS
Vol. 62, No. 3, November, 1955

Power-law Band Random Matrix

Wigner Band Random Matrix:
<u>Hamiltonians are real and symmetric</u>



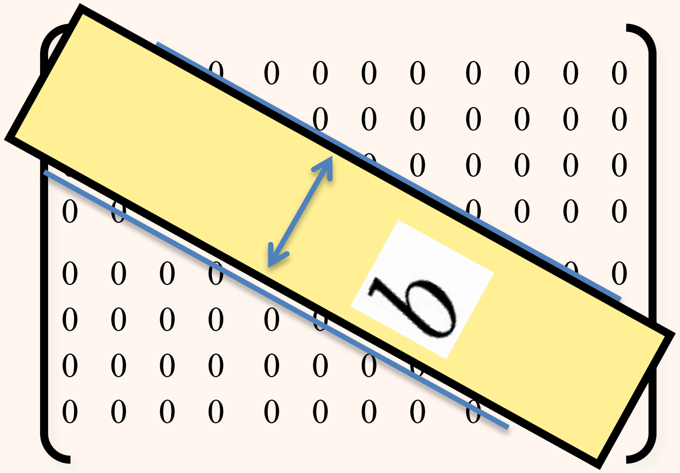
- Random real numbers
- Sparsity

Fyodorov, Casati, Izrailev, Prosen

OPEN QUESTION:
Which WBRM
(b, sparsity, correlations)
can represent well the realistic
spin models that we study?

Power-law Band Random Matrix

Wigner Band Random Matrix:
Hamiltonians are real and symmetric



- Random real numbers
- Sparsity

Fyodorov, Casati, Izrailev, Prosen

Power-law banded random matrix
Seligman, Kravtsov, Mirlin

$$\langle H_{nm} \rangle = 0$$

$$\langle H_{nn}^2 \rangle = 2$$

$$\langle H_{nm}^2 \rangle = \frac{1}{1 + \left| \frac{n-m}{b} \right|^2}$$

I. Varga, Phys. Rev. B **66**, 094201 (2002)

Two-Body Random Ensembles

Two-body random ensembles (1970)

French, Wong, Flores, Bohigas, Brody, Mello, Guhr, Weidenmüller, Izrailev, Flambaum, Kota, Zelevinsky, Horoi, Volya, Alhassid, Prosen, Seligman, …

$$H = \sum_k \varepsilon_k a_k^\dagger a_k + \lambda \sum_{k \le l, p \le q} \langle pq | V | kl \rangle a_p^\dagger a_q^\dagger a_l a_k,$$

↕ Embedded Random Ensembles

SYK model

Sachdev & Ye
PRL 70,3339 (1993)

We consider the ensemble of Hamiltonians

$$\mathcal{H} = \frac{1}{\sqrt{NM}} \sum_{i>j} J_{ij} \hat{\mathcal{S}}_i \cdot \hat{\mathcal{S}}_j, \tag{2}$$

where the sum over i, j extends over $N \to \infty$ sites, the exchange constants J_{ij} are mutually uncorrelated and selected with probability $P(J_{ij}) \sim \exp[-J_{ij}^2/(2J^2)]$, the $\hat{\mathcal{S}}$ are the spin operators of the group SU(M), and the states on each site belong to a representation labeled by the integer n_b [$n_b = 2S$ for SU(2); more generally n_b is the

More Realistic Random Matrices

Exercise BRM

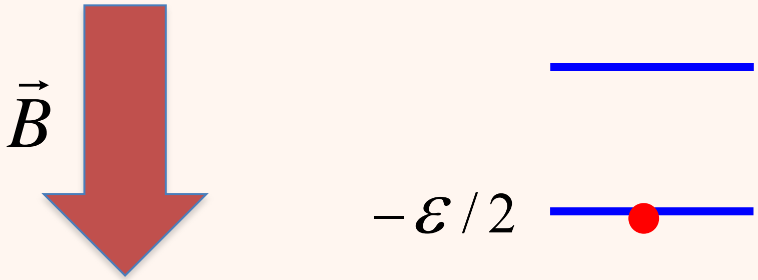
Choose one kind of random matrix
*) Wigner band random matrix with/without sparsity (vary the bandwidth)
*) Power-law banded random matrix (vary the *b*)
*) Two-body random ensembles
And compute everything we have seen so far:
*) Density of states
*) Level spacing distribution, level number variance
*) Entropies, PR for the eigenstates

FOR LATER:
*) Evolution of the survival probability
*) Evolution of the entropies, PR(t)

REALISTIC SYSTEMS
SPIN-1/2 MODELS

Two-Level System

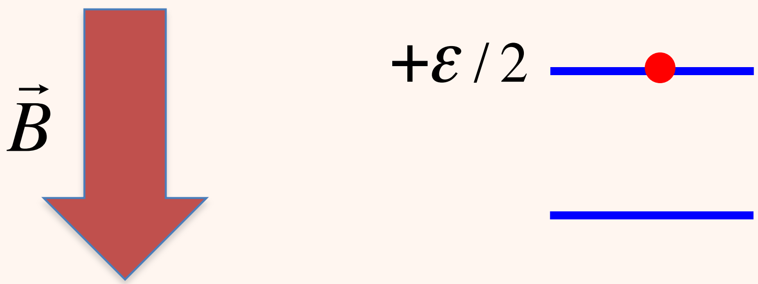
One spin-1/2



$$|0\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Two-Level System

One spin-1/2



$$|1\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

excitation

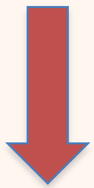
Two-Level System Hamiltonian

One spin-1/2

$$|1\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{———} \quad +\varepsilon/2$$

$$|0\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{———} \quad -\varepsilon/2$$

$\vec{B}$



Hamiltonian

$$H = \frac{\varepsilon}{2}\sigma^z$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} \frac{\varepsilon}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\varepsilon}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\varepsilon}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\varepsilon}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Ising Interaction

$$H = \frac{\varepsilon_1}{2}\sigma_1^z + \frac{\varepsilon_2}{2}\sigma_2^z + \frac{J\Delta}{4}\sigma_1^z\sigma_2^z$$

$$\frac{J\Delta}{4}\sigma_1^z\sigma_2^z|\uparrow\uparrow\rangle = +\frac{J\Delta}{4}|\uparrow\uparrow\rangle$$

$$\frac{J\Delta}{4}\sigma_1^z\sigma_2^z|\downarrow\downarrow\rangle = +\frac{J\Delta}{4}|\downarrow\downarrow\rangle$$

$$\frac{J\Delta}{4}\sigma_1^z\sigma_2^z|\uparrow\downarrow\rangle = -\frac{J\Delta}{4}|\uparrow\downarrow\rangle$$

$$\frac{J\Delta}{4}\sigma_1^z\sigma_2^z|\uparrow\downarrow\rangle = -\frac{J\Delta}{4}|\uparrow\downarrow\rangle$$

$$\sigma_1^z|\uparrow\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_2^z|\downarrow\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Flip-flop term

$$H = \frac{\varepsilon_1}{2}\sigma_1^z + \frac{\varepsilon_2}{2}\sigma_2^z + \frac{J\Delta}{4}\sigma_1^z\sigma_2^z + \frac{J}{4}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^x |\uparrow> = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow> \qquad \sigma^y |\uparrow> = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|\downarrow>$$

Flip-flop term

$$H = \frac{\varepsilon_1}{2}\sigma_1^z + \frac{\varepsilon_2}{2}\sigma_2^z + \frac{J\Delta}{4}\sigma_1^z\sigma_2^z + \frac{J}{4}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\frac{J}{4}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)|\uparrow\downarrow\rangle = \frac{J}{2}|\downarrow\uparrow\rangle$$

$$\frac{J}{4}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)|\downarrow\uparrow\rangle = \frac{J}{2}|\uparrow\downarrow\rangle$$

1D Spin-1/2 Systems

$$H=\sum_{n=1}^{L-1}\frac{J}{4}(\Delta\sigma_n^z\sigma_{n+1}^z+\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y)$$

Map into hardcore bosons:

$$H=\sum_{n=1}^{L-1}\left[V\left(b_n^+b_n-\frac{1}{2}\right)\left(b_{n+1}^+b_{n+1}-\frac{1}{2}\right)-t\left(b_n^+b_{n+1}+h.c.\right)\right]$$

Conserves the total magnetization in the z-direction

Dim is not 2^L but $\frac{L!}{up!(L-up)!}$

*) open/closed (open boundaries vs periodic boundaries)

*) isotropic $\Delta=1$ vs anisotropic $\Delta\neq 1$ XXX vs XXZ

*) nearest neighbor couplings

*) clean, homogenous vs disorder (onsite, couplings)

Hamiltonian Matrix: open XXZ

$\vert 1111\rangle$	$\vert 1110\rangle$	$\vert 1101\rangle$	$\vert 1011\rangle$	$\vert 0111\rangle$	$\vert 1100\rangle$	$\vert 1010\rangle$	$\vert 1001\rangle$	$\vert 0110\rangle$	$\vert 0101\rangle$	$\vert 0011\rangle$	$\vert 0001\rangle$	$\vert 0010\rangle$	$\vert 0100\rangle$	$\vert 1000\rangle$	$\vert 0000\rangle$
$\frac{3J\Delta}{4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	$+\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{J}{2}$	$-\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0	0	0	0	0	0	0	0
0	0	$\frac{J}{2}$	$-\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0	0	0	0	0	0	0
0	0	0	$+\frac{J}{2}$	$+\frac{J\Delta}{4}$	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	$+\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0	0	0	0	0
0	0	0	0	0	$\frac{J}{2}$	$-\frac{3J\Delta}{4}$	$\frac{J}{2}$	$\frac{J}{2}$	0	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{J}{2}$	$-\frac{J\Delta}{4}$	0	$\frac{J}{2}$	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{J}{2}$	0	$-\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0	0
0	0	0	0	0	0	0	$\frac{J}{2}$	$\frac{J}{2}$	$-\frac{3J\Delta}{4}$	$\frac{J}{2}$	0	0	0	0	0
0	0	0	0	0	0	0	0	0	$\frac{J}{2}$	$\frac{J\Delta}{4}$	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	$+\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0	0
0	0	0	0	0	0	0	0	0	0	0	$\frac{J}{2}$	$-\frac{J\Delta}{4}$	$\frac{J}{2}$	0	0
0	0	0	0	0	0	0	0	0	0	0	0	$\frac{J}{2}$	$-\frac{J\Delta}{4}$	$\frac{J}{2}$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{J}{2}$	$+\frac{J\Delta}{4}$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{3J\Delta}{4}$

Many-body system

$$H=\sum_{n=1}^{L-1}\frac{J}{4}(\Delta\sigma_n^z\sigma_{n+1}^z+\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y)$$

D: dimension of the Hilbert space

D = 2 x 2 x 2 x 2 x 2… = 2^L

0000

1000, 0100, 0010, 0001

1100, 1010, 1001, 0110, 0101, 0011

1110, 1101, 1001, 0011

1111

$$D=\binom{L}{L/2}=\frac{L!}{(L/2)!(L/2)!}$$

1100, 1010, 1001, 0110, 0101, 0011

BASIS:
Permutations of 111…000…

INTEGRABLE SPIN-1/2 MODEL

Integrable 1D Spin-1/2 Systems

Integrable system:

XXZ model

$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

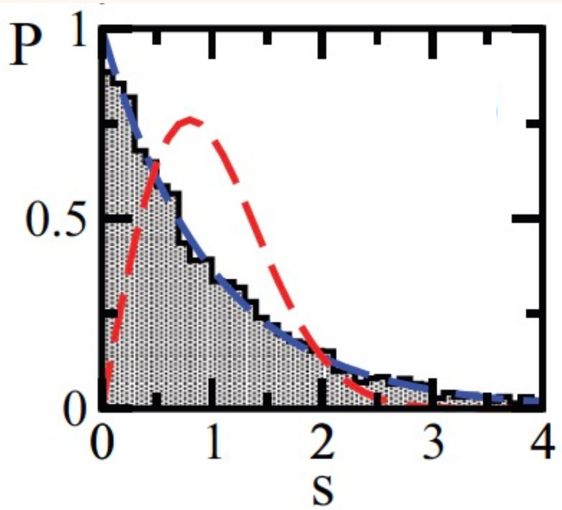
Solvable with the Bethe ansatz
See appendix of
ANNALS OF PHYSICS **182**, 280 (1988)
(coordinate Bethe ansatz)

$$\psi = \sum_{n=1}^{L} a(n)\phi(n)$$

$$\psi = \sum_{n,m=1}^{L} a(n,m)\phi(n,m)$$

Eigenvalues are uncorrelated

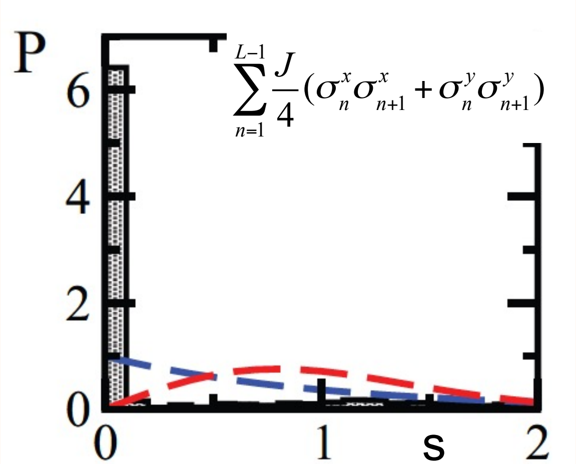
POISSON DISTRIBUTION



Integrable models

Integrable models:

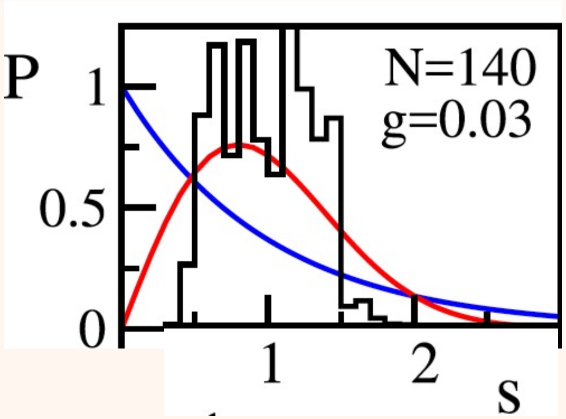
- Degeneracies



PRE**88,** 032913 (2013)

- Picket-fence spectrum

NJP**20**, 113039 (2018)



Wigner-Dyson Distribution and Integrability

- We can construct integrable Hamiltonians with WD distribution

Relaño, Dukelsky, Gómez, Retamosa
PRE **70**, 026208 (2004)

- Finite-size effect: Localization length is larger than the system size

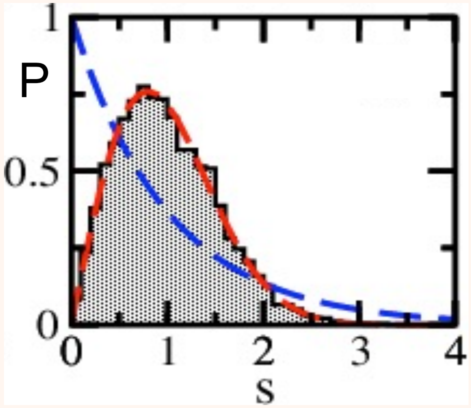
1D Anderson model

$$H = \sum_{n=1}^{L} \epsilon_n c_n^\dagger c_n - J \sum_{n=1}^{L-1} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$

1D Aubry-André model

$$H = \sum_{j=1}^{L} h \cos[(\sqrt{5}-1)\pi j + \phi] c_j^\dagger c_j - J \sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Tight-binding models



PRE**100**, 022142 (2019)

1D Anderson model: Level Repulsion

$$H = \sum_{n=1}^{L} \epsilon_n c_n^\dagger c_n - J \sum_{n=1}^{L-1} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$

$$[-W/2, W/2]$$

$$\xi = \frac{l_\infty(0)}{L} \approx \frac{105.045}{W^2 L}$$

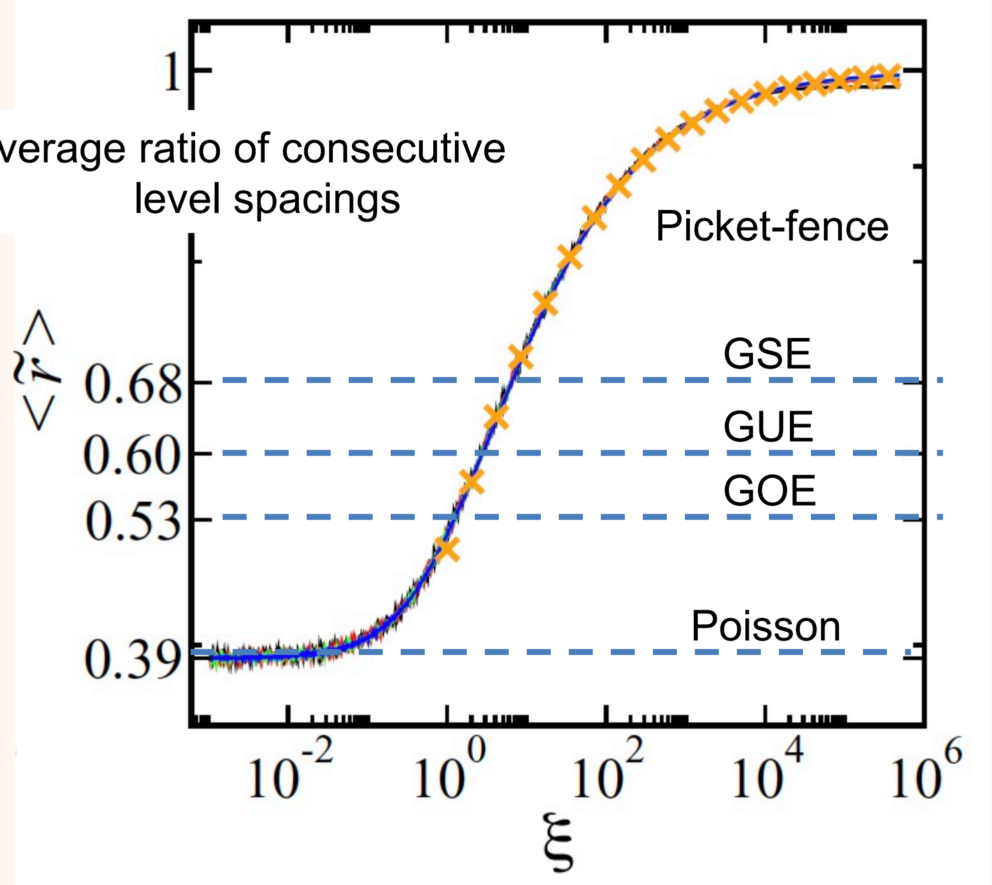
Tridiagonal matrix with random diagonal elements

Torres, Bermudez, LFS
PRE**100**, 022142 (2019)

Exercise 1D_Anderson

Show that we recover all degrees of level repulsion
-- Poisson, GOE, GUE, GSE, picket-fence --
by varying the parameter ξ

Average ratio of consecutive level spacings



Lea F. Santos, Yeshiva University

CHAOTIC SPIN-1/2 MODELS

1D Spin-1/2 Systems

Integrable system:

XXZ model

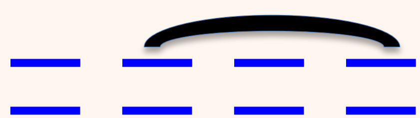
$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

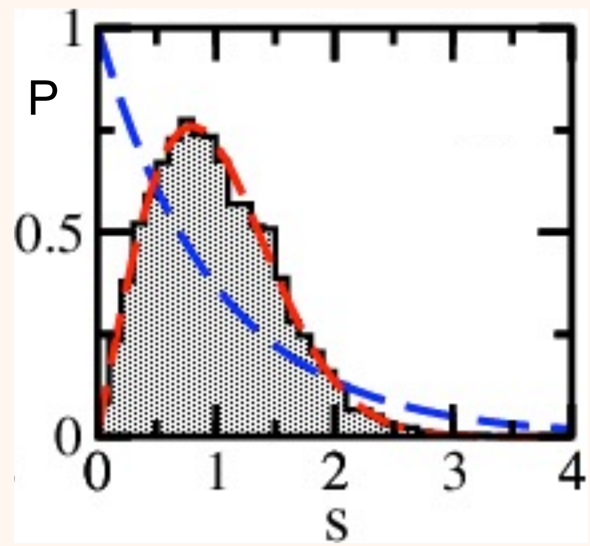
Interaction between next-nearest neighbors

WIGNER-DYSON DISTRIBUTION

$$H_{NN} + H_{NNN} = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \\ + \lambda \sum_{n=1}^{L-2} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+2}^z + \sigma_n^x \sigma_{n+2}^x + \sigma_n^y \sigma_{n+2}^y)$$

NNN model





1D Spin-1/2 Systems

Integrable system:

XXZ model

$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

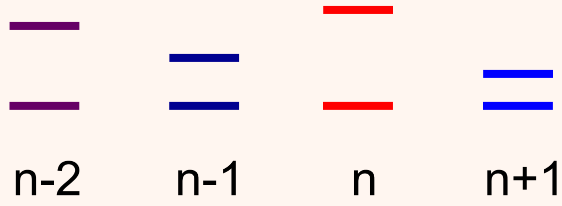
WIGNER-DYSON DISTRIBUTION

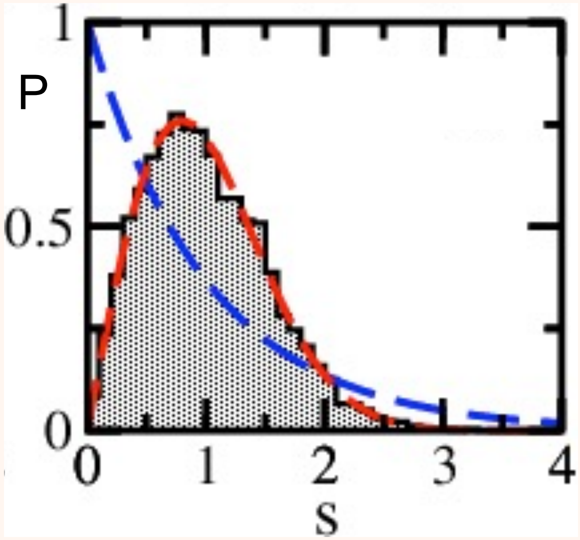
Many-body localization

$$H = \sum_{n=1}^{L} \frac{h_n}{2} \sigma_n^z + \sum_{n=1} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z)$$

Random numbers

$h_n \in [-h, h]$





1D Spin-1/2 Systems

Integrable system:

XXZ model

$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

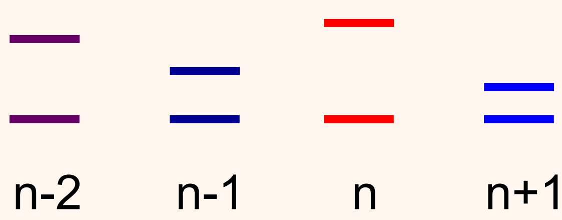
WIGNER-DYSON DISTRIBUTION

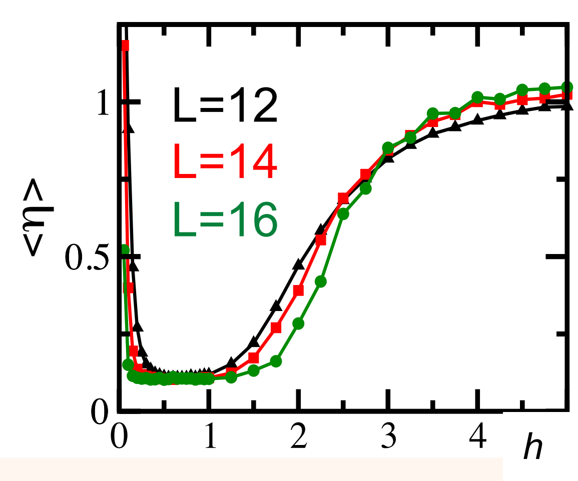
Many-body localization

$$H = \sum_{n=1}^{L} \frac{h_n}{2} \sigma_n^z + \sum_{n=1} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z)$$

Random numbers

$h_n \in [-h, h]$



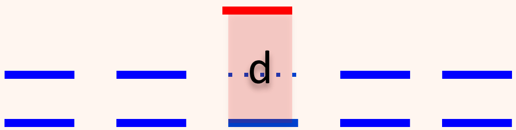


Lea F. Santos, Yeshiva University

Ann. Phys. **529**, 1600284 (2017)

Speck of Chaos

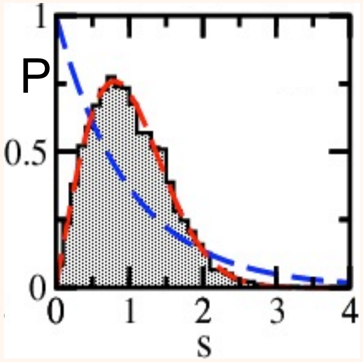
$$H_{one} = \frac{d_{L/2}}{2}\sigma^z_{L/2} + \sum_{n=1}^{L-1}\frac{J}{4}(\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + \Delta\sigma^z_n\sigma^z_{n+1})$$

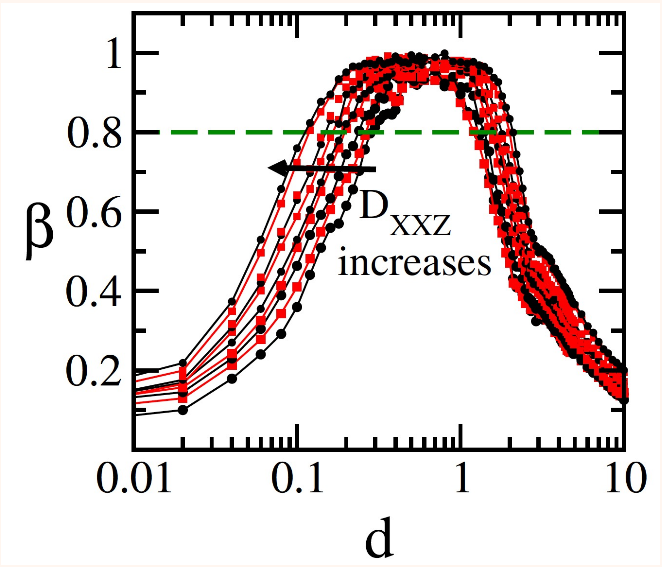


LFS,
JPA **37**, 4723 (2004)

$\beta \sim 1$ **chaos**

$$P_{WD}(s) = \frac{\pi s}{2}\exp\left(-\frac{\pi s^2}{4}\right)$$





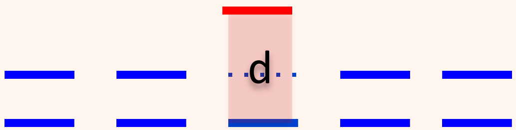
(local perturbation)

Gubin & LFS
AJP **80**, 246 (2012)

$$P(s) = (\beta+1)bs^{\beta}\exp(-bs^{\beta+1})$$

Speck of Chaos

$$H_{one} = \frac{d_{L/2}}{2}\sigma^z_{L/2} + \sum_{n=1}^{L-1}\frac{J}{4}\left(\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + \Delta\sigma^z_n\sigma^z_{n+1}\right)$$



LFS,
JPA **37**, 4723 (2004)

Chaos: Level statistics, chaotic eigenstates,
diagonal and off-diagonal elements of O

Chaos is the mechanism for **thermalization**

Chaos is the condition for the validity of ETH

Torres & LFS
PRE **89**, 062110 (2014)

Brenes, Mascarenhas, Rigol & Goold
PRB **98**, 235128 (2018)

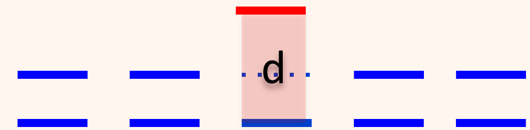
Ballistic quantum transport

M. Znidaric
PRL**125**, 180605 (2020)

Speck of Chaos
PRR **2**, 043034 (2020)
LFS, Bernal, Torres

Speck of Chaos

$$H_{XXZ} = d_{L/2} S^z_{L/2} + J \sum_{n=1}^{L-1} \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + \Delta S^z_n S^z_{n+1} \right)$$



LFS,
JPA **37**, 4723 (2004)

$$H_{ZZ} = d_{L/2} S^z_{L/2} + J h_x \sum_{n=1}^{L-1} S^x_n - J \sum_{n=1}^{L-1} S^z_n S^z_{n+1}$$

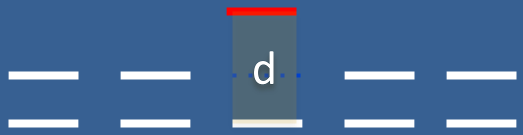
Ising model in a transverse field
Spin-1/2

$$H_{S1} = d_{L/2} S^z_{L/2} + J \sum_{n=1}^{L-1} \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + S^z_n S^z_{n+1} \right)$$
$$+ J \sum_{n=1}^{L-1} \left((S^x_n S^x_{n+1})^2 + (S^y_n S^y_{n+1})^2 + (S^z_n S^z_{n+1})^2 \right)$$

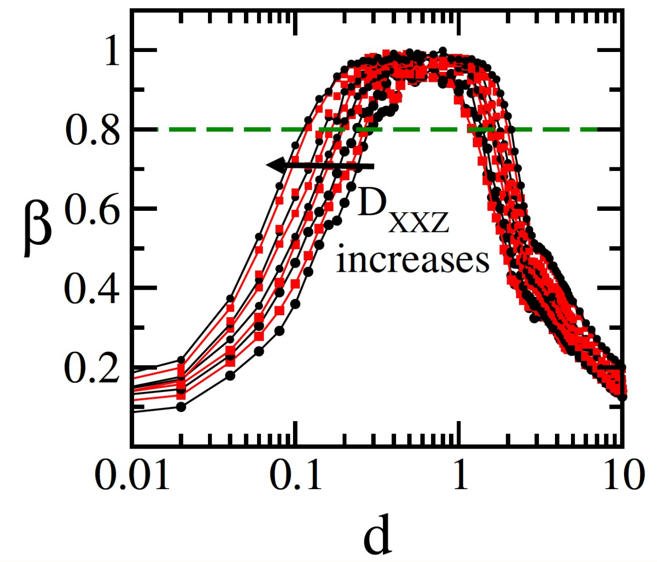
Lai-Sutherland model
Spin-1

Speck of Chaos
PRR **2**, 043034 (2020)
LFS, Bernal, Torres

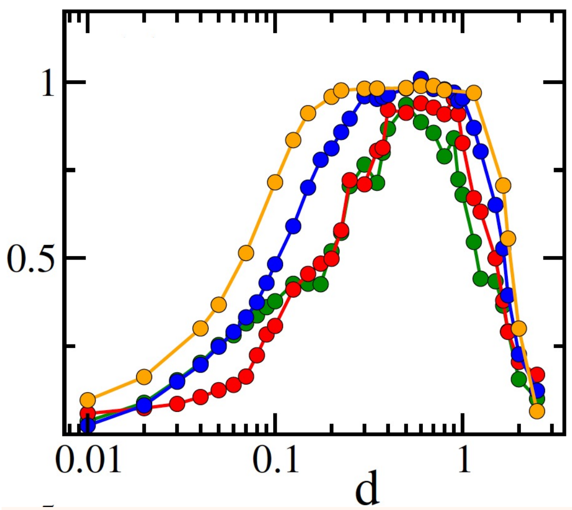
Speck of Chaos



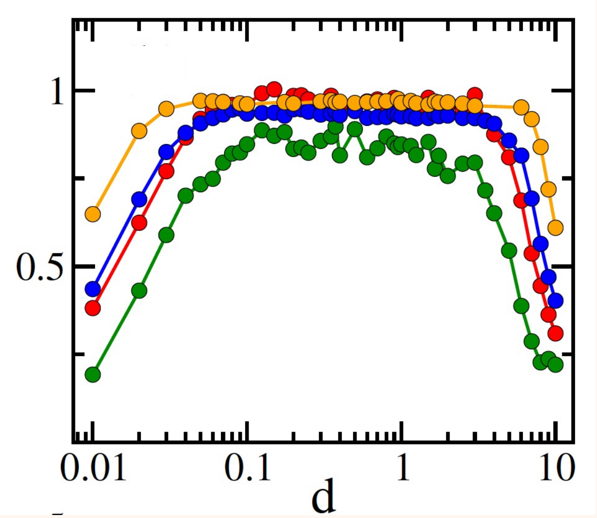
XXZ + defect
Spin-1/2

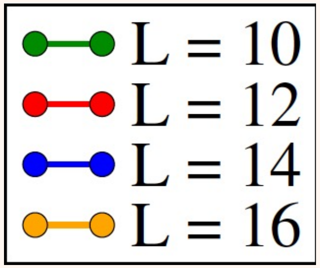


Ising + defect
in a transverse field
Spin-1/2



Lai-Sutherland + defect
Spin-1





$$P(s) = (\beta + 1) b s^{\beta} \exp(-b s^{\beta+1})$$

$\beta \sim 1$ **chaos**

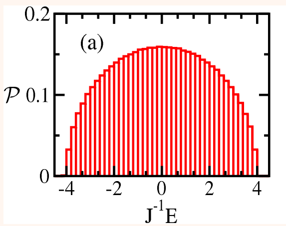
Speck of Chaos
PRR **2**, 043034 (2020)
LFS, Bernal, Torres

DOS

of systems with
2-body couplings and
many excitations
(integrable or chaotic)

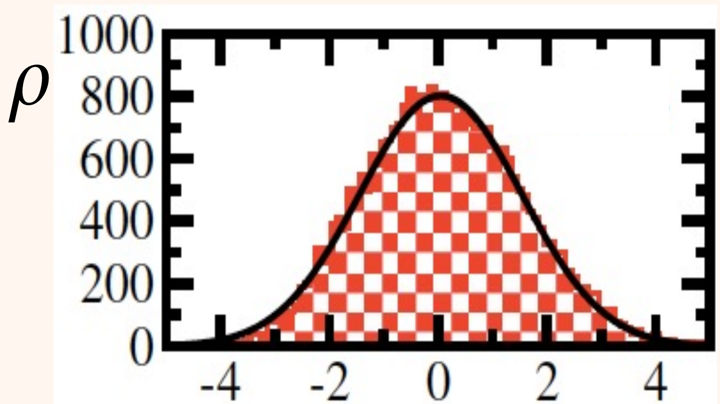
GAUSSIAN SHAPE

Gaussian DOS



Wigner (1957)

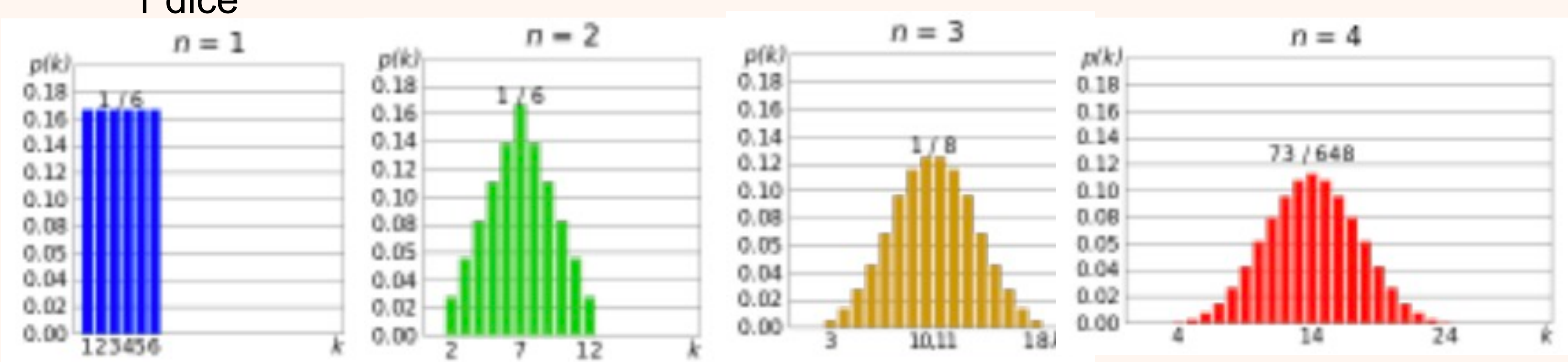
Many-body quantum systems with two-body interactions: Gaussian



French & Wong, PLB (1970) E_α

1 dice

4 dice



From few- to many-body quantum systems

XX model
$$H_{XX} = \sum_{n=1}^{L-1} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

Just 1 excitation:
$$|\psi_\alpha\rangle = \sum_{l=1}^{L} a_l^\alpha |\phi_l\rangle$$

$$|\phi_1\rangle = |\uparrow \downarrow \downarrow \cdots \downarrow\rangle$$
$$|\phi_2\rangle = |\downarrow \uparrow \downarrow \cdots \downarrow\rangle$$
$$\cdots .$$

$$H|\psi_\alpha\rangle = E_\alpha |\psi_\alpha\rangle$$

$$H|\phi_1\rangle = \frac{J}{2}(|\phi_L\rangle + |\phi_2\rangle);$$
$$H|\phi_l\rangle = \frac{J}{2}(|\phi_{l-1}\rangle + |\phi_{l+1}\rangle), \quad l \neq 1, L$$
$$H|\phi_L\rangle = \frac{J}{2}(|\phi_{L-1}\rangle + |\phi_1\rangle).$$

Now collect the terms with the same index *l*

Lea F. Santos, Yeshiva University

Quantum Sci. Technol. **3**, 044006 (2018)

From few- to many-body quantum systems

Just 1 excitation:

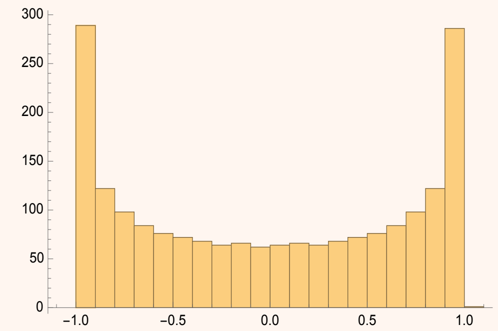
$$|\psi_\alpha\rangle = \sum_{l=1}^{L} a_l^\alpha |\phi_l\rangle$$

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$$

$$H|\phi_1\rangle = \frac{J}{2}(|\phi_L\rangle + |\phi_2\rangle);$$
$$H|\phi_l\rangle = \frac{J}{2}(|\phi_{l-1}\rangle + |\phi_{l+1}\rangle), \quad l \neq 1, L$$
$$H|\phi_L\rangle = \frac{J}{2}(|\phi_{L-1}\rangle + |\phi_1\rangle).$$

Collecting the terms with the same index l:

$$E_\alpha a_l^\alpha = \frac{J}{2}(a_{l-1}^\alpha + a_{l+1}^\alpha)$$



Ansatz: $a_l^\alpha = e^{i\theta l}$ ⟹ $E_\alpha = J\cos\theta$

Periodic boundary conditions: $a_{l+L} = a_l$ ⟹ $\theta = \frac{2\pi k}{L}$

$$k \in \left\{0, \pm 1, \pm 2, \cdots, \pm\left(\frac{L}{2} - 1\right), \frac{L}{2}\right\}$$

Lea F. Santos, Yeshiva University

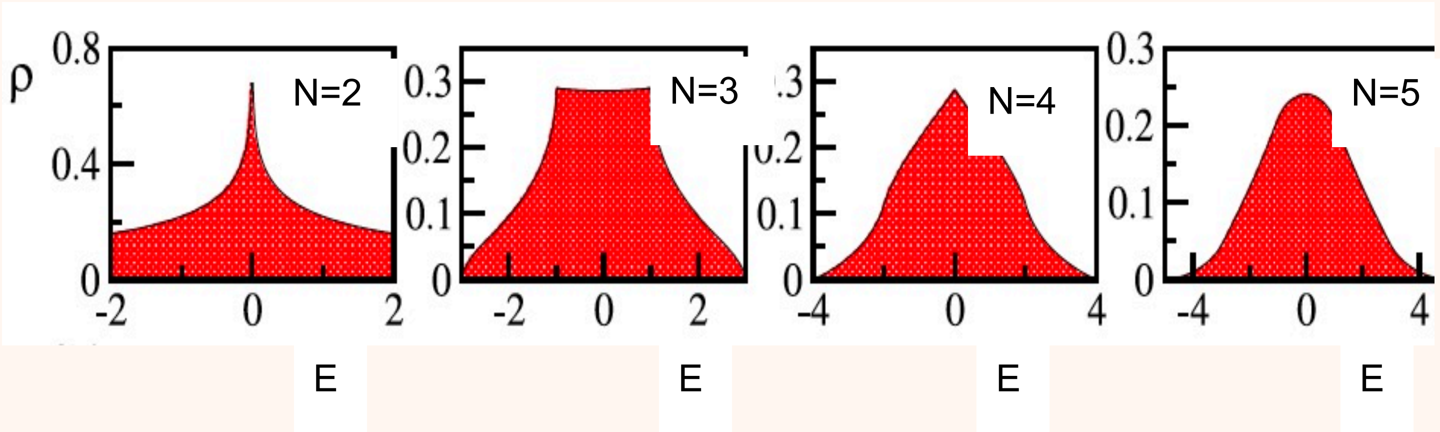
From few- to many-body quantum systems

XX model $$H_{XX} = \sum_{n=1}^{L-1} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

$$E_\alpha = J \sum_{i=1}^{N} \cos\left(\frac{2\pi k_i}{L}\right)$$

$$k_1 < k_2 < \cdots < k_N$$

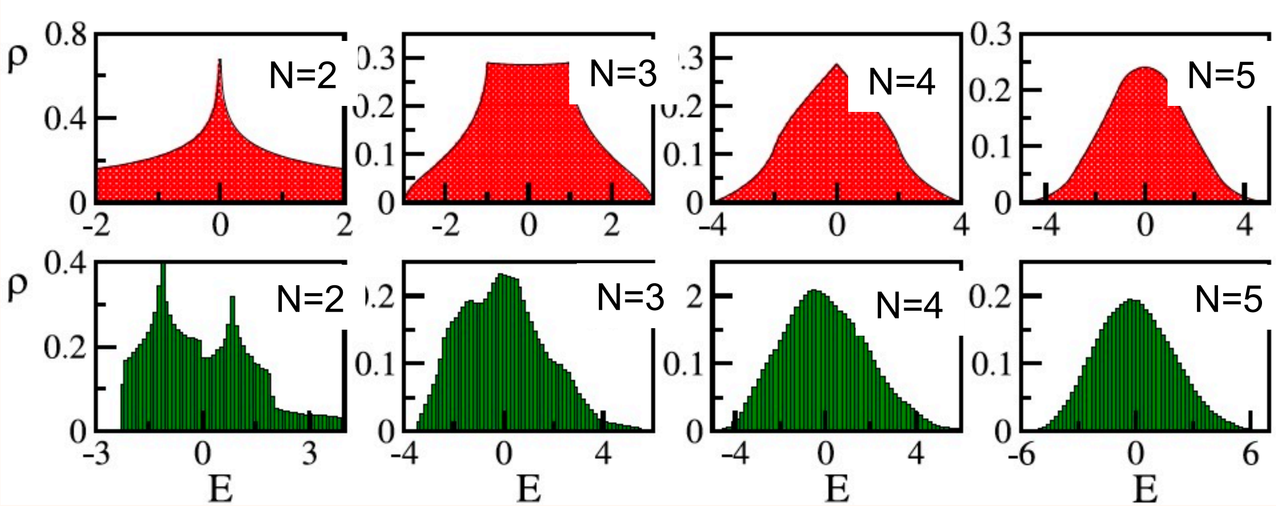
$$k_i \in \{0, \pm 1, \pm 2, \cdots, \pm(L/2 - 1), L/2\}$$



From few- to many-body quantum systems

XX model

$$H_{XX} = \sum_{n=1}^{L-1} \frac{J}{4}(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$



NN+NNN model

$$H_{NN} + H_{NNN} = \sum_{n=1}^{L-1} \frac{J}{4}(\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \lambda \sum_{n=1}^{L-2} \frac{J}{4}(\Delta \sigma_n^z \sigma_{n+2}^z + \sigma_n^x \sigma_{n+2}^x + \sigma_n^y \sigma_{n+2}^y)$$

Lea F. Santos, Yeshiva University

Quantum Sci. Technol. **3**, 044006 (2018)

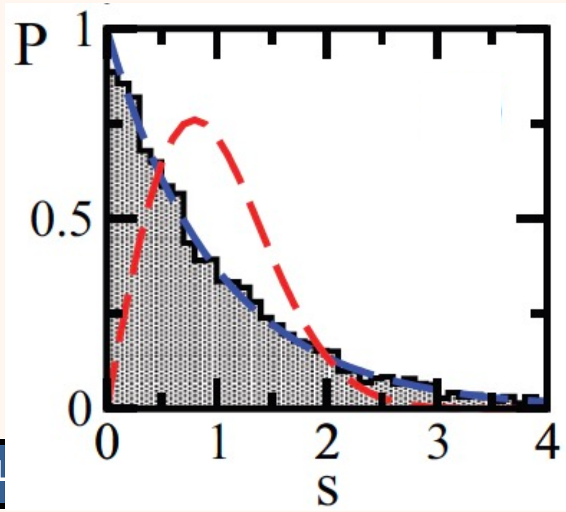
1D Spin-1/2 Systems

Integrable system:

XXZ model

$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

POISSON DISTRIBUTION

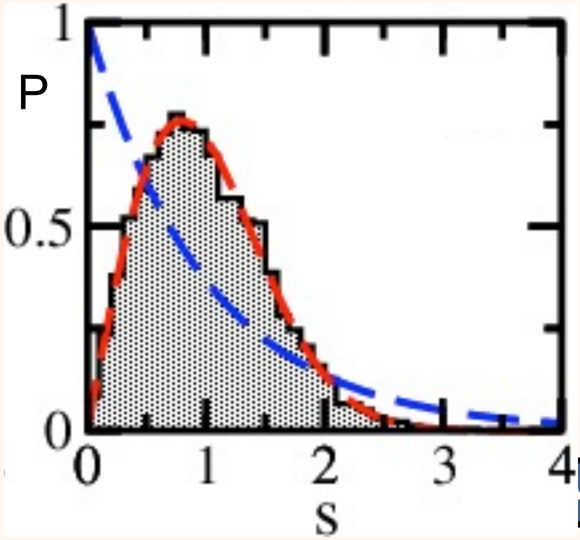


Chaotic model:

Couplings between 2nd neighbors

$$H = \sum_{n=1}^{L-1} \frac{J}{4} (\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$
$$+ \lambda \sum_{n=1}^{L-2} \frac{J}{4} (\Delta \boldsymbol{\sigma}_n^z \boldsymbol{\sigma}_{n+2}^z + \boldsymbol{\sigma}_n^x \boldsymbol{\sigma}_{n+2}^x + \boldsymbol{\sigma}_n^y \boldsymbol{\sigma}_{n+2}^y)$$

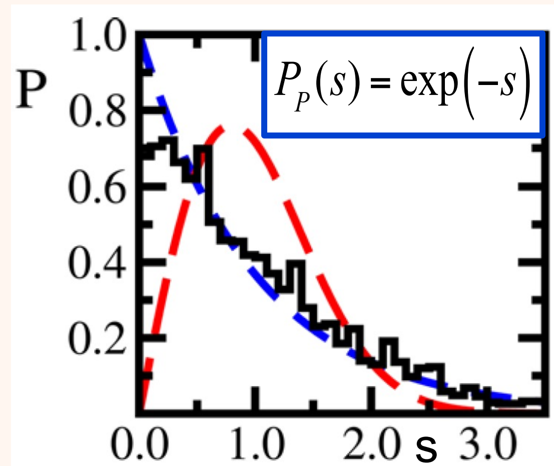
WIGNER-DYSON DISTRIBUTION



Chaotic Models and Poisson distribution

- **Chaotic** models but mixed symmetries:

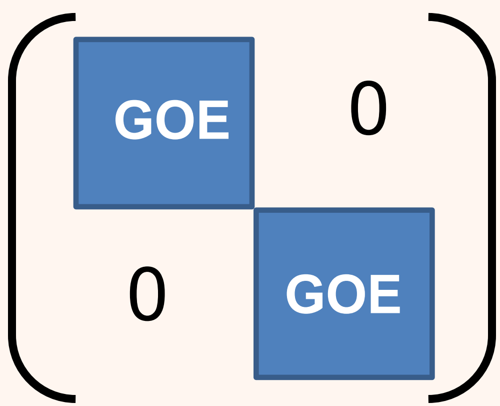
$$H_{NN} + H_{NNN} = \sum_{n=1}^{L-1} \frac{J}{4}(\Delta \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) +$$
$$+\lambda \sum_{n=1}^{L-2} \frac{J}{4}(\Delta \sigma_n^z \sigma_{n+2}^z + \sigma_n^x \sigma_{n+2}^x + \sigma_n^y \sigma_{n+2}^y)$$

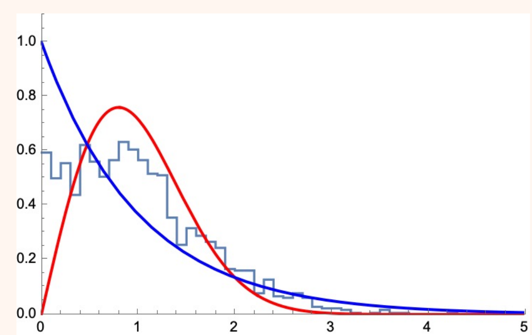


AJP**80**, 246 (2012)

Conservation of total spin: $\Delta = 1$

Parity: clean

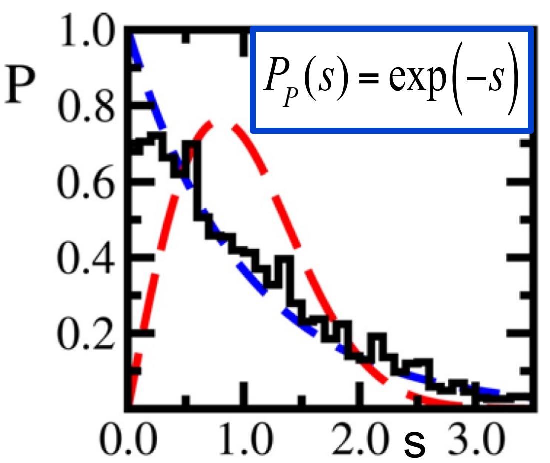




Chaotic Models and Poisson distribution

- **Chaotic** models but mixed symmetries:

$$H_{NN} + H_{NNN} = \sum_{n=1}^{L-1} \frac{J}{4}(\Delta\sigma_n^z\sigma_{n+1}^z + \sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y) +$$
$$+\lambda\sum_{n=1}^{L-2} \frac{J}{4}(\Delta\sigma_n^z\sigma_{n+2}^z + \sigma_n^x\sigma_{n+2}^x + \sigma_n^y\sigma_{n+2}^y)$$



AJP**80**, 246 (2012)

Two options:
*) Take symmetries into account
or
*) Avoid symmetries

Avoiding symmetries:

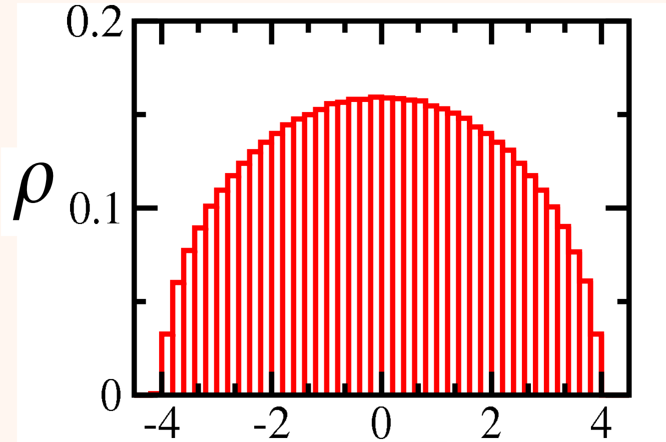
*) $\Delta \neq 1$

*) Add a defect: $0.1J\sigma_1^z$

Full Random Matrices vs Two-Body Interaction

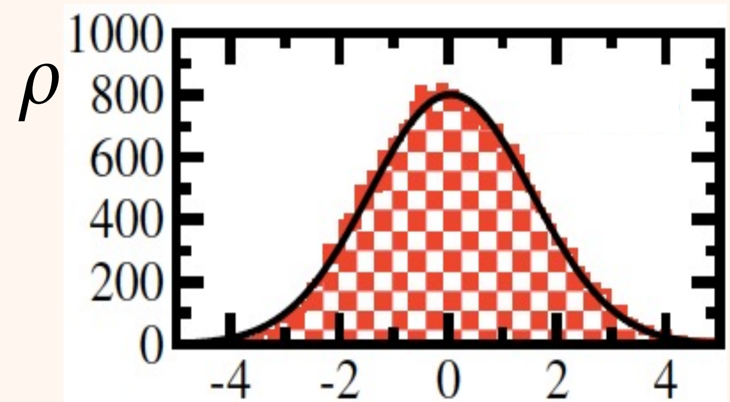
Density of States (Energy Distribution)

Full random matrices: semicircular



Wigner (1957)

Two-body interactions: Gaussian

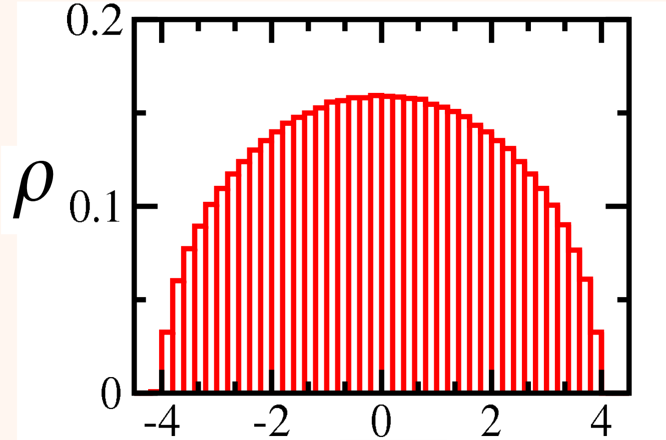


French & Wong, PLB (1970) E_α

Full Random Matrices vs Two-Body Interaction

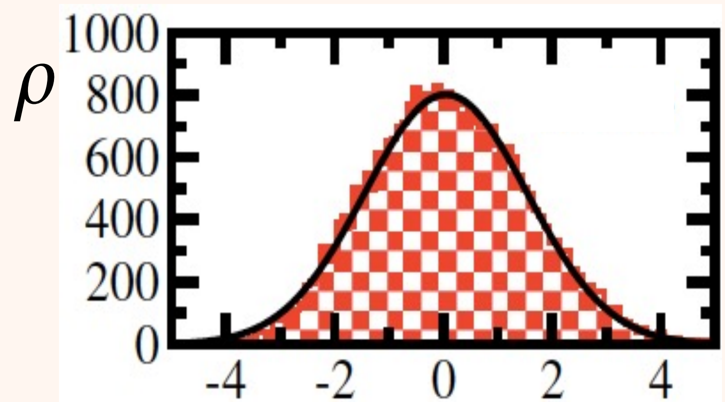
Density of States (Energy Distribution)

Full random matrices: semicircular

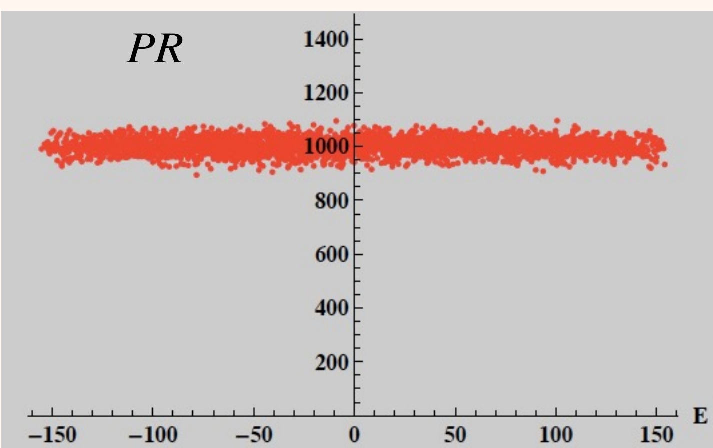


Wigner (1957)

Two-body interactions: Gaussian



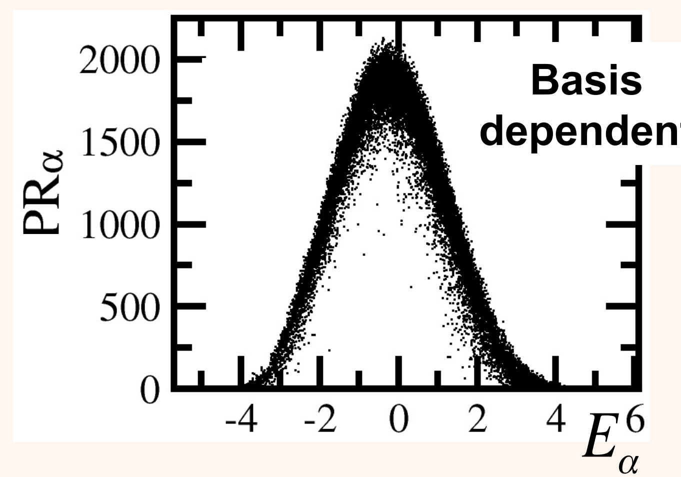
French & Wong, PLB (1970) E_α



Participation Ratio

$$PR^{(\alpha)} \equiv \frac{1}{\sum_{i=1}^{D} |c_i^{(\alpha)}|^4}$$

$$\psi^{(\alpha)} = \sum_{i=1}^{D} c_i^{(\alpha)} \phi_i$$



Measures of delocalization: Spin models

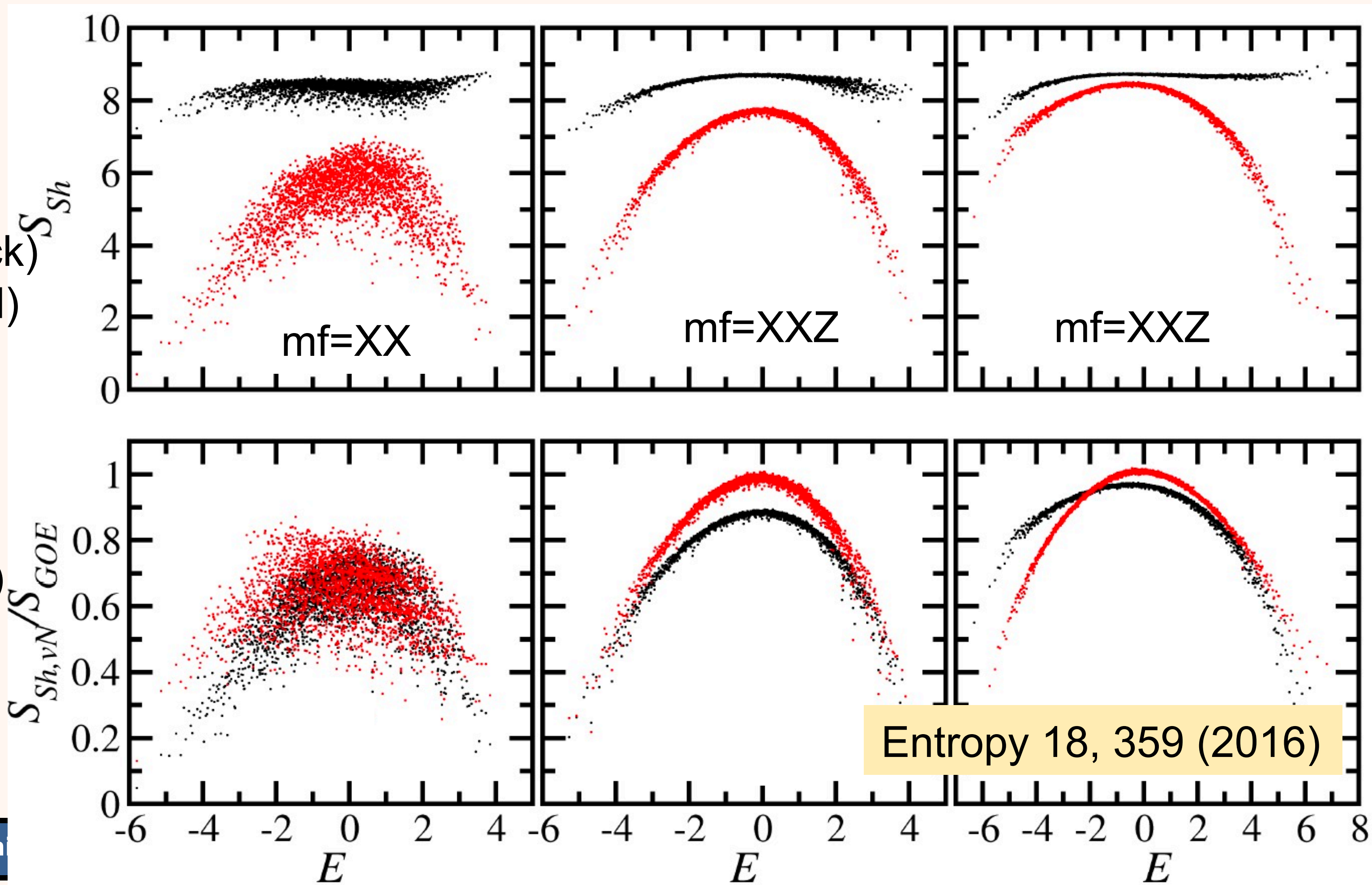
INTEGRABLE — Clean XXZ

CHAOTIC — XXZ + middle defect — Clean NN+NNN

Shannon entropy in the site-basis (black) mean-field (red)

Shannon entropy mf (red)

Entanglement entropy (black)



Entropy 18, 359 (2016)

Exercise SpinHamiltonian

Write the Hamiltonians for the spin-1/2 chains

*) XXZ (open)

*) XXZ+ single-defect (open)

*) XXZ + NNN (open)

*) XXX + onsite disorder (closed)

*) Ising +single defect (open)

Compute DOS, P(s), rtilde

*) PR, entropies in site-basis

*) PR, entropies in mean-field basis

Thermalization

$$\langle O(t)\rangle = \langle \Psi(t) | O | \Psi(t)\rangle = \sum_{\alpha\neq\beta} C_\beta^{ini*} C_\alpha^{ini} e^{i(E_\beta - E_\alpha)t} O_{\beta\alpha} + \sum_\alpha | C_\alpha^{ini} |^2 O_{\alpha\alpha}$$

$$|\Psi(0)\rangle = \sum_\alpha C_\alpha^{ini} |\alpha\rangle \qquad O_{\beta\alpha} = \langle \beta | O | \alpha \rangle$$

Quantum chaos and thermalization in isolated systems of interacting particles
Borgonovi, Izrailev, LFS, Zelevinsky
Physics Reports **626**, 1 (2016)

From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics,
L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol,
Adv. Phys. **65**, 239 (2016)

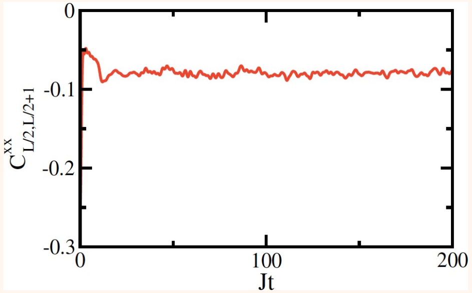
Thermalization

$$\langle O(t)\rangle = \langle \Psi(t)|O|\Psi(t)\rangle = \sum_{\alpha\neq\beta} C_\beta^{ini*} C_\alpha^{ini} e^{i(E_\beta - E_\alpha)t} O_{\beta\alpha} + \sum_\alpha |C_\alpha^{ini}|^2 O_{\alpha\alpha}$$

$$|\Psi(0)\rangle = \sum_\alpha C_\alpha^{ini} |\alpha\rangle$$

$$O_{\beta\alpha} = \langle \beta|O|\alpha\rangle$$

Equilibration:



Size of the fluctuations
PRE **88**, 032913 (2013)

Components are small and **uncorrelated**
Lack of degeneracies: eigenvalues are **correlated**
Off-diagonal elements of local observables are small

Thermalization

$$\langle O(t)\rangle = \langle \Psi(t)|O|\Psi(t)\rangle = \sum_{\alpha\neq\beta} C_\beta^{ini*} C_\alpha^{ini} e^{i(E_\beta - E_\alpha)t} O_{\beta\alpha} + \sum_\alpha |C_\alpha^{ini}|^2 O_{\alpha\alpha}$$

Infinite time average Thermodynamic average

$$\overline{\langle O(t)\rangle} \equiv \sum_\alpha |C_\alpha^{ini}|^2 O_{\alpha\alpha} \overset{=?}{\longleftrightarrow} O_{micro} \equiv \frac{1}{\mathcal{N}_{E_0,\Delta E}} \sum_{\substack{\alpha \\ |E_0 - E_\alpha| < \Delta E}} O_{\alpha\alpha}$$

$$\langle \alpha | O | \alpha \rangle$$

depends on the initial conditions depends only on the energy

ETH: the expectation values $O_{\alpha\alpha}$ of few-body observables do not fluctuate for eigenstates close in energy

Chaos guarantees thermalization, ETH

Chaotic states

Thermalization in realistic systems happens away from the edges of the spectrum

Lea F. Santos, Yeshiva University

Observables

Spin-spin correlations in the μ-direction

$$C^{\mu\mu}_{L/2,L/2+1} = \langle \hat{S}^{\mu}_{L/2} \hat{S}^{\mu}_{L/2+1} \rangle$$

Structure factor: Fourier transform of the spin-spin correlations

$$\hat{S}^{\mu\mu}(k) = \frac{1}{L} \sum_{l,j=1}^{L} \hat{S}^{\mu}_{l} \hat{S}^{\mu}_{j} e^{-ik(l-j)}$$

LFS & M. Rigol
PRE **81** (2010)
PRE **82** (2010)
M. Rigol and LFS
PRA **82** (R) (2010)
Torres & LFS
PRE **88** (2013)
PRE **89**, 062110 (2014)

Peres Lattice

PRL **53**, 1711 (1984)

$$PR^{(\alpha)} = 1 / \sum_{i=1}^{D} |c_i^{(\alpha)}|^4$$

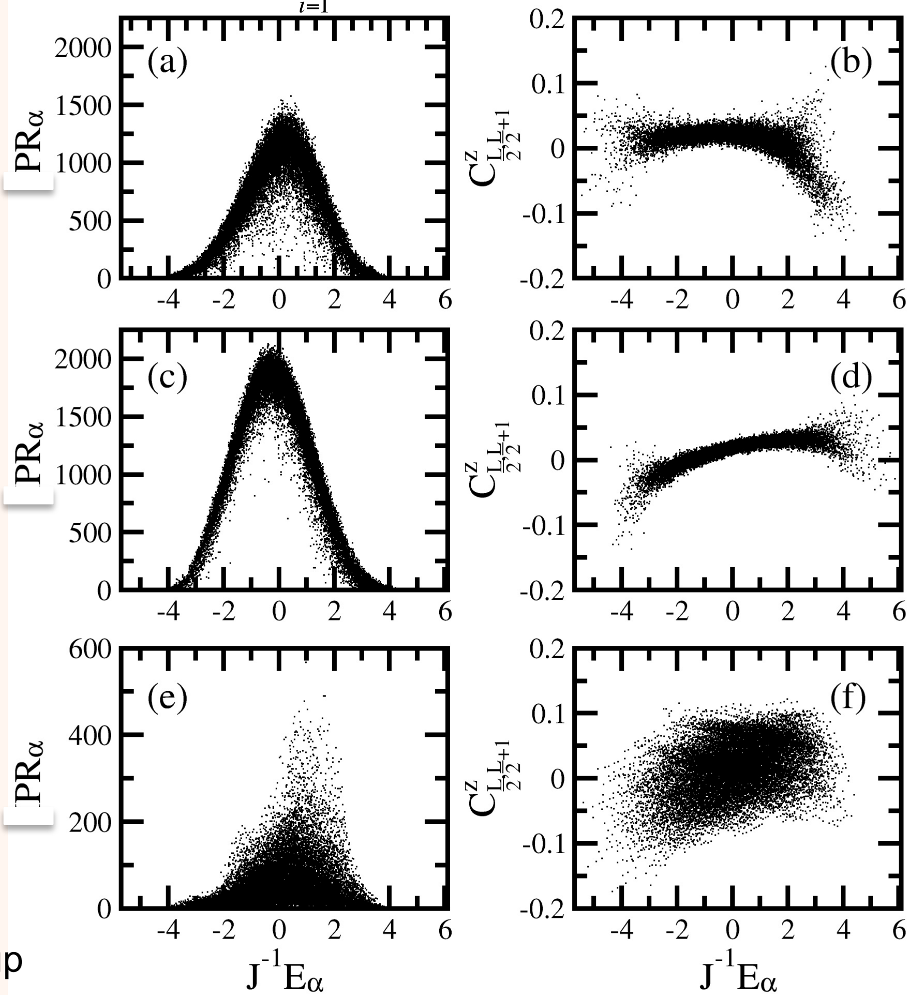
$$O_{\alpha\alpha} = \langle \alpha | O | \alpha \rangle$$

$$\Theta O = (O_{max} - O_{min}) / O_{micro}$$

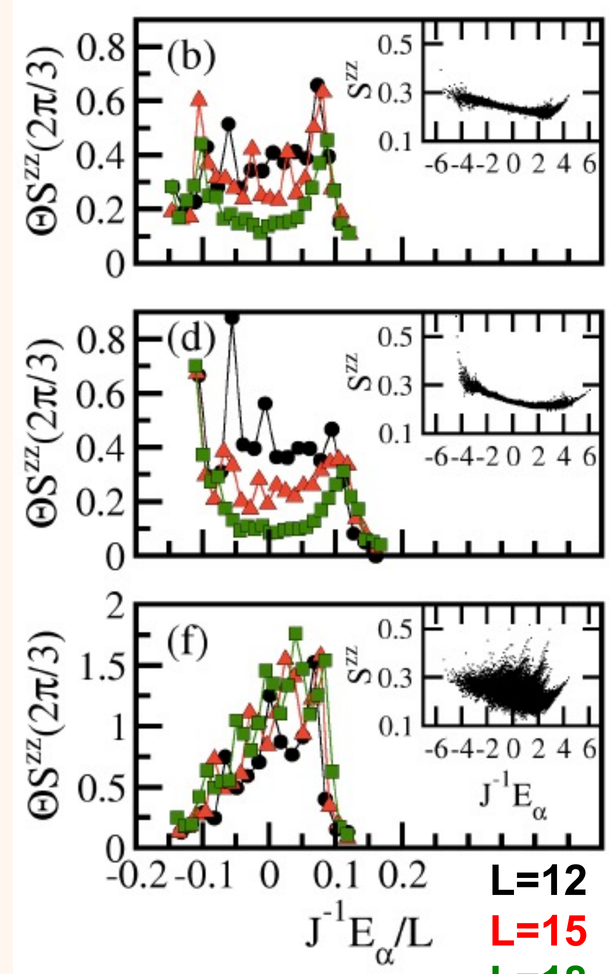
Chaotic Single-Defect Model

Chaotic NNN Model

Integrable XXZ Model



L=18, 1/3 up



PRE **89**, 062110 (2014)

EEV and Microcanonical Average

Eigenstate expectation values (EEVs) vs microcanonical average

$$\Delta^{\mathrm{mic}} O \equiv \frac{\sum_\alpha |O_{\alpha\alpha} - O_{\mathrm{mic}}|}{\sum_\alpha O_{\alpha\alpha}}$$

where

$$O_{\mathrm{mic}} = \frac{1}{\mathcal{N}_{E,\Delta E}} \sum_{\substack{\alpha \\ |E-E_\alpha|<\Delta E}} O_{\alpha\alpha}$$

Normalized extremal fluctuation

$$\Delta_e^{\mathrm{mic}} O \equiv \left| \frac{\max O - \min O}{O_{\mathrm{mic}}} \right|$$

LFS & M. Rigol
PRE **81** (2010)
PRE **82** (2010)
M. Rigol and LFS
PRA **82** (R) (2010)
Torres & LFS
PRE **88** (2013)
PRE **89**, 062110 (2014)

LFS & M. Rigol
PRE **82** (2010)

Infinite-time Average and Microcanonical Average

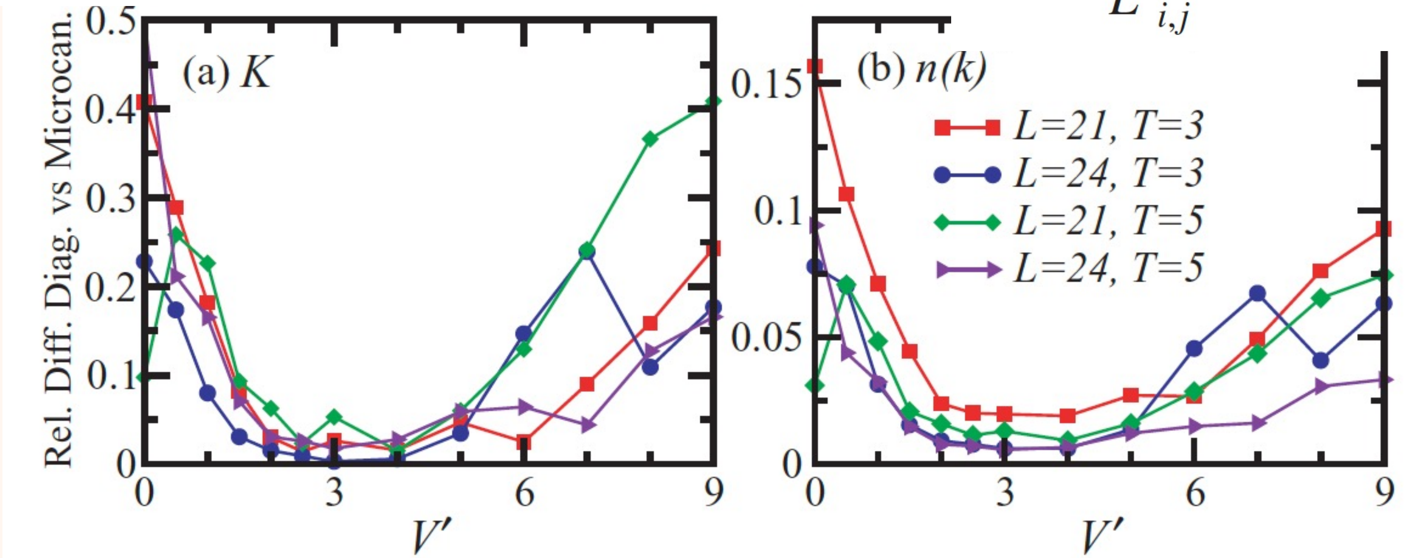
Proximity between the
infinite-time average (diagonal ensemble) and **microcanonical average**

Kinetic energy

$$[K = \sum_i -t(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.})]$$

Momentum distribution function

$$\hat{n}(k) = \frac{1}{L}\sum_{i,j} e^{-k(i-j)}\hat{b}_i^\dagger \hat{b}_j$$



$$\hat{H}_b = \sum_{i=1}^{L}\left\{-t(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}) + V\left(\hat{n}_i^b - \frac{1}{2}\right)\left(\hat{n}_{i+1}^b - \frac{1}{2}\right) + V'\left(\hat{n}_i^b - \frac{1}{2}\right)\left(\hat{n}_{i+2}^b - \frac{1}{2}\right)\right\}, \quad (1)$$

M. Rigol and LFS
PRA **82** (R) (2010)

Thermalization

$$\langle O(t)\rangle = \langle \Psi(t) | O | \Psi(t)\rangle = \sum_{\alpha\neq\beta} C_\beta^{ini*} C_\alpha^{ini} e^{i(E_\beta - E_\alpha)t} O_{\beta\alpha} + \sum_\alpha | C_\alpha^{ini} |^2 O_{\alpha\alpha}$$

$$O_{\beta\alpha} = \langle \beta | O | \alpha \rangle$$

Quantum chaos and thermalization in isolated systems of interacting particles
Borgonovi, Izrailev, LFS, Zelevinsky
Physics Reports **626**, 1 (2016)

From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics,
L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol,
Adv. Phys. **65**, 239 (2016)

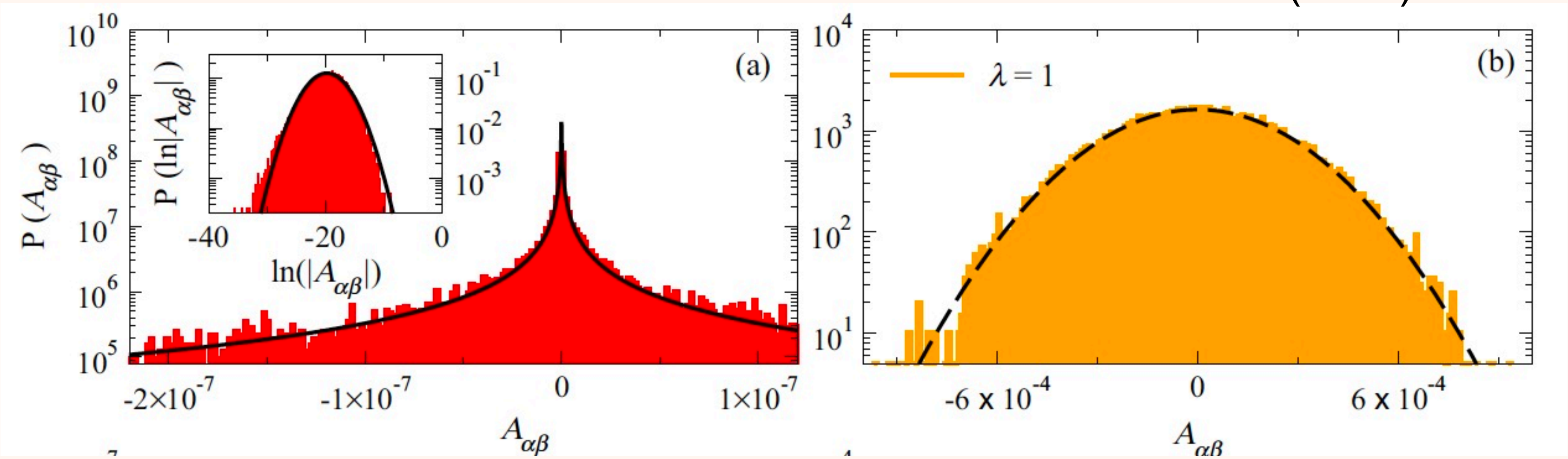
Off-diagonal elements

Beugeling, Moessner, Haque
PRE **91**, 012144 (2015)
(Gaussian distribution)

$$O_{\beta\alpha} = \langle \beta | O | \alpha \rangle$$

Integrable model (XXZ)

Chaotic model (NNN)



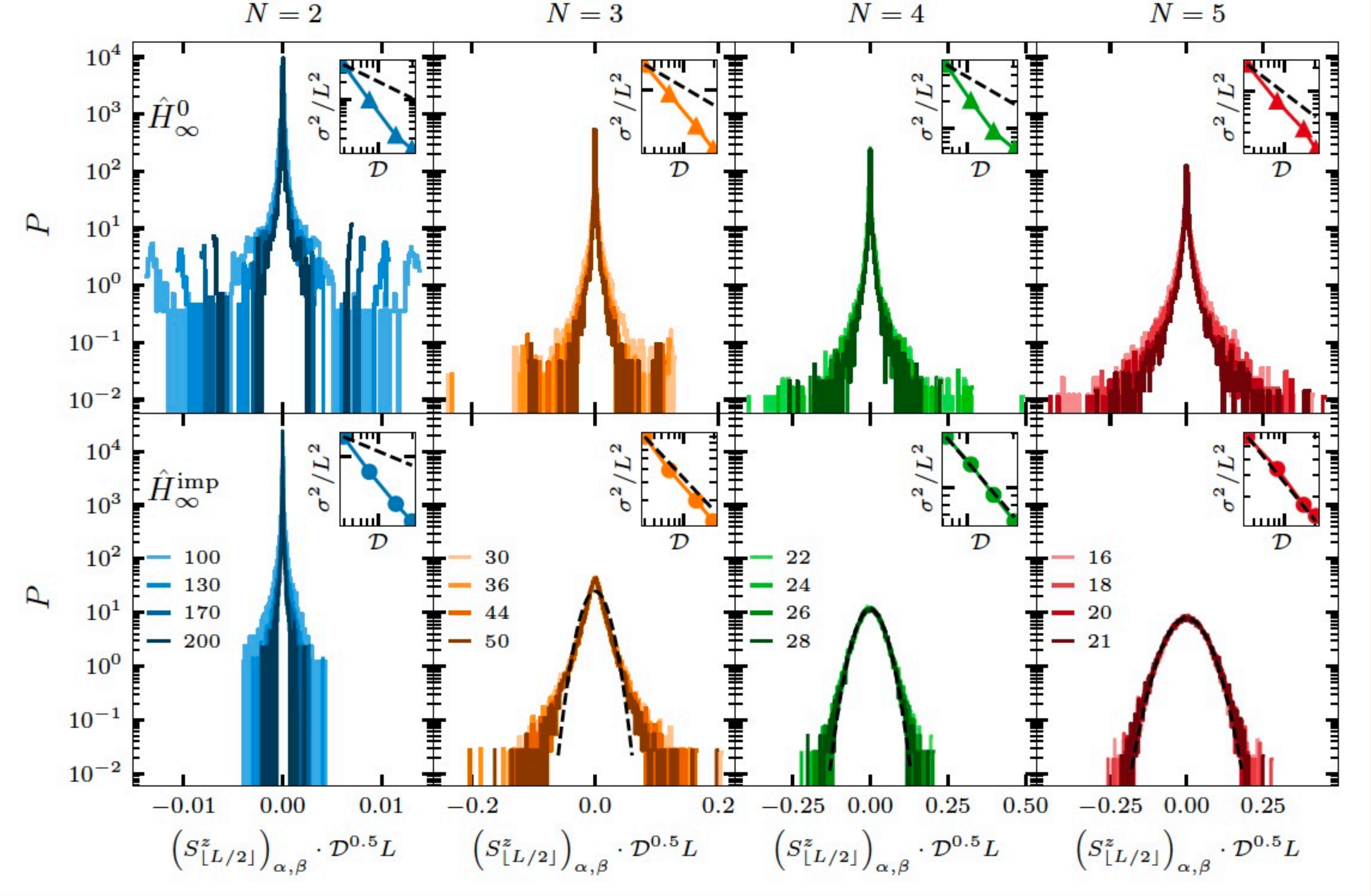
LeBlond, Mallayya, Vidmar, Rigol
PRE **100**, 062134 (2019)

As the number of excitations increases
Off-diagonal elements

$O_{\beta\alpha} = \langle \beta | O | \alpha \rangle$

Clean XXZ

XXZ + single defect



Lea F. Santos, Yeshiva University

SciPost Phys. **10**, 088 (2021)

Defect models
Off-diagonal elements

$$O_{\beta\alpha} = \langle \beta | O | \alpha \rangle$$

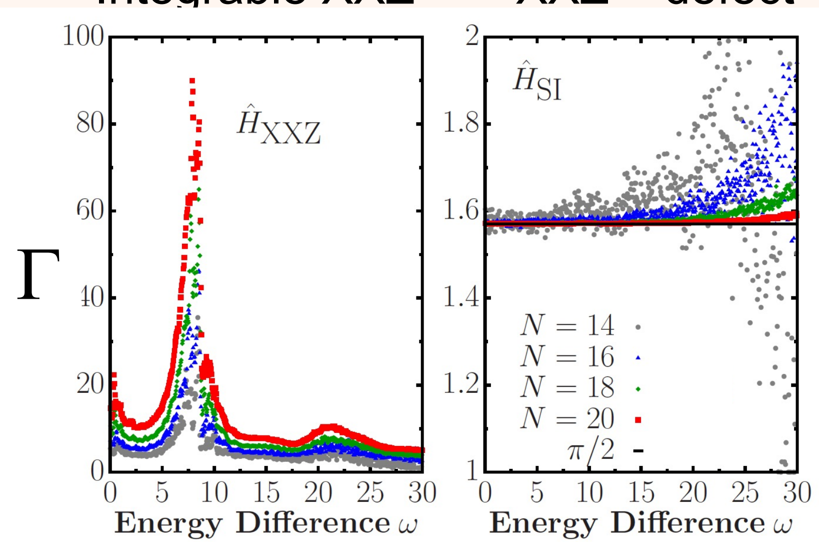
$$\Gamma(\omega) = \frac{\overline{|\langle \alpha | O | \beta \rangle|^2}}{\overline{|\langle \alpha | O | \beta \rangle|}^2}$$

$\Gamma(\omega)=\pi/2$

means Gaussian distribution

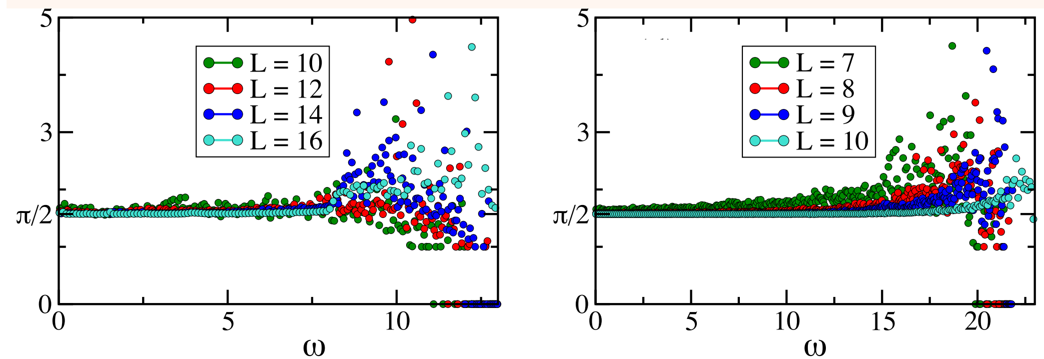
Beugeling, Moessner, Haque
PRE **91**, 012144 (2015)
(Gaussian distribution)

Integrable XXZ   XXZ + defect



Brenen, Goold, Rigol
PRB **102**, 075127 (2020)

Ising + defect   Spin 1 + defect



Speck of Chaos
PRR **2**, 043034 (2020)
LFS, Bernal, Torres

Off-diagonal elements and symmetries

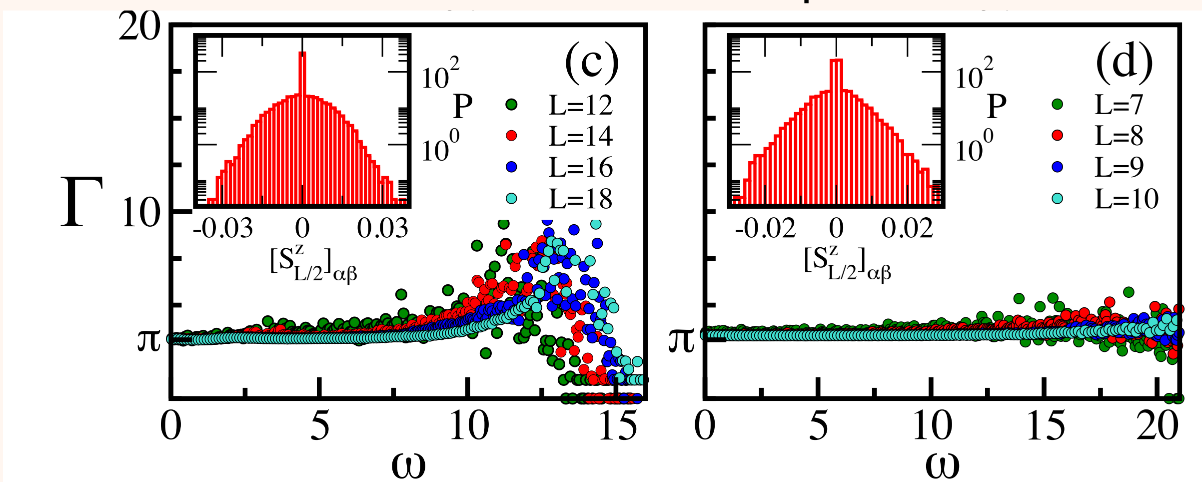
$$\Gamma(\omega) = \overline{\left|\langle \alpha | S^z_{L/2} | \beta \rangle\right|^2} \Big/ \left|\overline{\langle \alpha | S^z_{L/2} | \beta \rangle}\right|^2$$

OPEN QUESTION

No need for unfolding
Detect chaos despite symmetries

NN+NNN

Spin 1 + border



Speck of Chaos
PRR **2**, 043034 (2020)
LFS, Bernal, Torres

Exercise Thermalization

*) Use the XXZ + λ NNN (open, λ=1) model with a small defect at the edge and analyze diagonal and off-diagonal elements of physical observables.
*) Choose some observables. See examples on slide 75.
*) Study EEV vs E.
EEV = eigenstate expectation value
*) Study the quantities on slides 79.
*) Study the distribution of the off-diagonal elements (choose ~200 eigenstates in the middle of the spectrum).
*) Compute kurtosis vs λ